On Fourth-Order Green's Function for Euler-Bernoulli Beam Equation

By
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This thesis, written by **Ebrahim Saleh Mohammed Bakalah** under the direction of his thesis advisor and approved by his thesis committee, has been presented and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of

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*Dedicated To **My Mother**, Father, Wife, Brothers All Family.*
To My Friends.
To All, Who Waited Patiently For Me To Come Out Of My Study And For Their Continuous Support, Endless Love And Encouragement.

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"**In The Name Of Allah, Most Compassionate, Most Merciful**"

All praises are for Allah, the highest, most gracious and most merciful. May his peace, blessings and mercy be upon his noble messenger and Prophet Mohammed, his family, his companions and those who follow their footsteps till the last day.

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BVP	:	Boundary Value Problem
S-LP	:	Sturm-Liouville Problem
Eq.	:	Equation

ABSTRACT

Full Name : Ebrahim Saleh Mohammed Bakalah

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In this thesis, we consider the static and dynamic problems arising from Euler-Bernoulli beam model with inhomogeneous elastic properties. The method of Green's function and perturbation theory are employed to find the deflection in the beam correct to the first-order. The eigenvalue problems arising from the transverse vibrations of inhomogeneous beams in linear cases have been considered and a non-linear case has also been considered.

ملخص الرسالة

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هذه الرسالة تدرس المشاكل الإستاتيكية والديناميكية الناتجة عن نموذج أُيلر- برنولي للجسور ذات الخصائص المرنة غير المتجانسة. لدراسة هذه المشاكل فقد تم إستخدام طريقة معادلة جرين ونظرية الإضطراب من أجل حساب مقدار الإنحراف الحاصل في الجسر محسوبة إلى الدرجة الأولى من التقريب. إضافة إلى ذلك، فقد تمت دراسة مسألة قيم الإيجن الناتجة عن الإهتزازات العرضية الحاصلة في الجسور ذات الخصائص غير المتجانسة الخطية منها والغير خطية.

CHAPTER 1

MATHEMATICAL PRELIMINARIES

In this chapter, we introduce some basic concepts and notions together with mathematical results required in our study.

1.1 Introduction and Description of Results

The beams and girders are extensively used in civil and mechanical engineering. One of the earliest models is Euler-Bernoulli model, to study the bending of beams it has been based upon small deflections of a beam subjected to lateral loads only. One may refers to Truesdell [17] for account of development of this approach in 1750. It was a little consequence in terms of applications till it became a cornerstone of engineering in the late of 19th century. A recent work Han, Benaroya and Wei [7] have given a good account of different models of elastic beams including the Euler-Bernoulli beam. The use of spectral properties, Green's function and perturbation method has been an important tools in second-order problems arising from vibration, elastic, acoustic and electromagnetic waves. One may find a description of these methods in Stakgold and Holst [14], Lindell and Olyslager [8] and Logan [9]. The inhomogeneity in the medium has been dealt by a number of authors by using the perturbation approach. To mention a few, Ghosh [3] used this approach to study Love waves in an inhomogeneous medium. Subsequently, Zaman *et al* used the Green function coupled with perturbation method to discuss field due to a point source in an inhomogeneous medium [2], [19], dispersion of Love waves in a stochastic layer [18] and inverse scattering in multilayer inverse problem [20]. Stuwe and Werner [15] used Green's function to study potential flow in infinite cylindrical channels. Gupta [6] studied existence and uniqueness for fourth-order equation arising from bending of an elastic beam. Graef and Yang [5] discussed this problem for non-linear load and established existence results using the Green function approach. More recently, Graef, Henderson and Yang [4] discussed positive solutions of fourth-order problems using this approach. Orucoglu [11] has also used the Green function approach to deal with a completely non-homogenous boundary value problem. More recently, Pietramala [13] used the Green function to study the beam equation with non-linear boundary conditions. Furthermore, Palamides, Panos K. and Alex P. Palamides [12] studied four-point fourth-order boundary value problems. Teterina [16] has obtained the relevant results for the Green function of the fourth-order differential linear operator defined by $\frac{d^4(-)}{dx^4}$ and used it to solve some boundary value problems. Morrison [10] in Master's thesis has applied Green's function to solve third-order non-linear boundary value problems. Abu-Hilal [1] used Green's function to study forced vibration of Euler-Bernoulli beam in case of different homogenous and elastic boundary conditions for dynamic response due to distributed or concentrated loads.

In this thesis, our study here will be different in two aspects, we consider the fourth-order static and dynamic problems arising from homogenous and non-homogenous elastic properties of Euler-Bernoulli beam model in linear and non-linear cases. We obtain Green's function for fourth-order differential linear operator for the following three types of boundary conditions:

- Hinged boundary conditions (simply supported beam).
- Clamped boundary conditions (fixed supported beam).
- Clamped-Free boundary conditions (cantilever beam).

The Green function is then used to study perturbed static and dynamic problems for non-homogenous beam. Our study here is different for the earlier works in two aspects. We shall consider static as well as dynamic problems arising from the Euler-Bernoulli beam equation. More important, we consider the beam to be non-homogenous having variable elastic properties.

In Chapter 1, we present a brief review of literature, some mathematical preliminaries and methodology. The Euler-Bernoulli beam equations for static and dynamic problems are also given in this Chapter.

In Chapter 2, the Green function for the fourth-order Euler-Bernoulli beam equation has been obtained for the previous three types of boundary conditions (hinged boundary conditions, clamped boundary conditions and clamped –free boundary conditions) that occur frequently in practice. The deflection of the beam under external force is obtained using the Green function.

Chapter 3 is devoted to the perturbation problem arising from beams with non-homogenous elastic properties. The deflection in this case is obtained regarding to some boundary conditions (hinged boundary conditions, clamped boundary conditions and clamped –free boundary conditions) and graphically represented for two interest materials cases one of them is for concrete beam and the other one is for steel beam.

In Chapter 4, perturbed eigenvalue problem is studied arising from vibration of non-homogenous beams. The unperturbed non-linear problem is discussed and a perturbed non-linear problem is also discussed regarding to the boundary conditions (hinged boundary conditions, clamped boundary conditions and clamped –free boundary conditions).

In Chapter 5, given conclusions and recommendations for future work and for further studies.

1.2 Euler-Bernoulli Beam Equation of Homogenous Elastic Properties

The transverse bending and vibration of beams are one of the fundamental problems in civil and mechanical engineering.

In the static case, the Euler-Bernoulli beam model leads to a fourth-order ordinary differential equation with appropriate boundary conditions imposed at the ends. However, the vibrating beam results in a model in terms of a fourth-order partial differential equation with initial and boundary conditions specified. In the Euler-Bernoulli model, the deflection of the beam is considering in the transverse direction with rotational effects are assumed negligible.

Many structures are constructed using beams or girders. These beams have distortion or deflection under their weights or some external forces. We shall see that the deflection in the beam can be found by solving a fourth-order differential equation.

To start, let us suppose that a homogenous beam with length has a uniform rectangular cross section without any load on the beam (except its weight). Consider the line that in the axis of symmetry is in the horizontal direction as shown in Fig. 1.1.

Assume that a load acts in the beam in the vertical direction due to the weight of the beam. In our model, we assume that the line through the axis of symmetry deflects connecting to the same points as shown in the following Fig. 1.1:

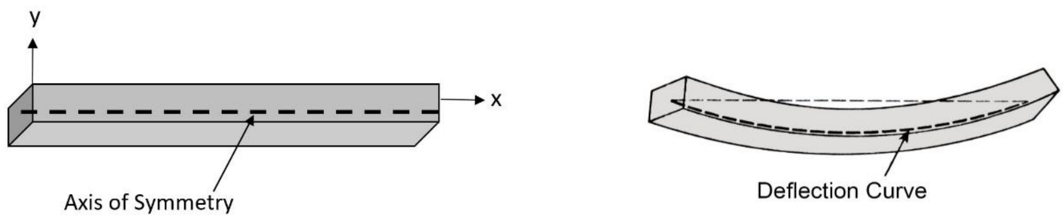

Figure 1.1: ***Deflection of a homogeneous beam***

From the theory of elasticity, it can be observed that the moment of the bending $M(x)$ at a point (say x) over the beam is related to the load $f(x)$ is given by following equation:

$$\frac{d^2M}{dx^2} = f(x). \tag{1.1}$$

Furthermore, the moment of bending $M(x)$ is proportional to the curvature k (say) of the deflected curve, i.e.

$$M(x) = EIk, \tag{1.2}$$

where E is the Young's modulus of elasticity and I is the moment of inertia of the beam cross section. The constant EI is called the flexural rigidity of the beam.

We know from calculus the curvature is given by:

$$k = \frac{u''}{[1+(u')^2]^{2/3}}. \tag{1.3}$$

If the deflection tangential slope $u(x)$ is assumed to be small, so that:

$$1+(u')^2 \approx 1. \tag{1.4}$$

Hence the curvature is given by $k = u''$, then the Equation (1.2) leads to the following:

$$M = EI \frac{d^2u}{dx^2}. \tag{1.5}$$

If we differentiate Equation (1.5) twice with respect to x, we get:

$$\frac{d^2M}{dx^2} = \frac{d^2}{dx^2}(EI \frac{d^2u}{dx^2}) = EI \frac{d^4u}{dx^4}. \tag{1.6}$$

By substituting Equation (1.1) in Equation (1.6), we have that the deflection $u(x)$ satisfies the following fourth-order differential equation:

$$EI \frac{d^4u}{dx^4} = f(x). \tag{1.7}$$

This is so-called the Euler-Bernoulli beam equation for homogenous elastic beam.

1.2.1 Boundary Conditions

The boundary conditions of beam depend upon how the beam is held at the ends as following:

i. Clamped End (Fixed Supported End)

If the beam is clamped at the end $x = a$, then there will be no deflection change in the tangential slope at that end and so we have:

$$u(a) = 0,\ u'(a) = 0.$$

ii. Free End

If the end at $x = a$ is free, then the bending moment and the shear force are zero and therefore we have:

$$u''(a) = 0,\ u'''(a) = 0.$$

iii. Hinged End (Simply Supported End)

If the end at $x = a$ is simply supported, then the deflection and the bending moment are zero and therefore we have:

$$u(a) = 0,\ u''(a) = 0.$$

As example, in case of a cantilever beam, one end is clamped (at zero) while the other end is free (at one), so that we can have the following boundary conditions for a cantilever beam:

$$u(0) = 0,\ u'(0) = 0, \qquad \text{at clamped end}$$

$$u''(0) = 0,\ u'''(0) = 0. \qquad \text{at free end.}$$

Assume we have a beam with length $l = [a, b]$ under its unit, where $a, b \in R$ and $0 < a < b$.The following table summarizes the boundary conditions at the ends of the beam listed as hinged boundary conditions (simply supported beam), clamped boundary conditions (fixed supported beam) and clamped-free boundary conditions (cantilever beam) that correspond to the previous pictures in Fig. 1.1.

Table 1.1: ***Boundary conditions of the beam shown in Fig. 1.1***

Ends of the beam	Boundary conditions at the ends $x=a,\ x=b$
i. Hinged (simply supported) beam	$u(a)=0,\ u''(a)=0,\ u(b)=0,\ u''(b)=0$
ii. Clamped (fixed supported) beam	$u(a)=0,\ u'(a)=0,\ u(b)=0,\ u'(b)=0$
iii. Clamped-Free (cantilever) beam	$u(a)=0,\ u'(a)=0,\ u''(b)=0,\ u'''(b)=0$

1.2.2 Dynamic Case (Vibration of Euler-Bernoulli Beam Equation)

Let us consider a beam with a rectangular cross section. Let $M(x,t)$ be the bending moment, $V(x,t)$ be the shear force and $f(x,t)$ be the external force per unit length of the beam. Then the inertia force acting on an element of beam is:

$$\rho A(x)\frac{\partial^2 w}{\partial t^2},$$

where $A(x)$ is the area of cross section of the beam, w is the transverse displacement.

The resultant force in z direction gives:

$$-(V+dV)+f(x,t)dx+V=\rho A(x)\frac{\partial^2 w}{\partial t^2},$$

where ρ is the mass density.

The moment of force about $y-axis$ is:

$$(M + dM) - (V + dV)dx + \frac{1}{2} f(x,t)dx - M = 0.$$

Using $dV = \frac{\partial V}{\partial x} dx$, $dM = \frac{\partial M}{\partial x} dx$ and ignoring terms of higher power of dx, we get after rearrangement:

$$\frac{\partial^2 M}{\partial x^2} + f(x,t) = \rho A(x) \frac{\partial^2 w}{\partial t^2}.$$

As $M(x,t) = EI(x) \frac{\partial^2 w}{\partial x^2}$, we get:

$$c^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0, \qquad (*)$$

where $c = \sqrt{\frac{EI}{\rho A}}$.

This is the dynamic equation of Euler-Bernoulli beam. To seek solutions for transverse vibrations of the beam equation that are time harmonic, we assume:

$$w(x,t) = W(x)e^{iwt},$$

where w is the angular frequency of vibrations. Using this, the Equation (*) becomes:

$$\frac{\partial^4 W}{\partial x^4} = \beta W(x),$$

where $\beta = \frac{w^2}{c^2}$.

1.3 Second-Order Boundary Value Problem

The discussion about the beam is not withstanding, although we are interested in dealing with the fourth-order beam equation, it is useful to have the results available for the corresponding second-order differential equation.

The second-order boundary value problem arises in a number of situations such as wave propagation, heat or diffusion process. We consider the following boundary value problem:

$$a(x)\frac{d^2u}{dx^2}+b(x)\frac{du}{dx}+c(x)u=f(x), \qquad a<x<b, \tag{1.8}$$

where $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are continuous functions on $[a,b]$, $a(x)\neq 0 \ \ \forall x$ on the interval $[a,b]$.

The boundary conditions are:

$$\begin{aligned} A_1u(a)+B_1u(a)&=C_1,\\ A_2u(b)+B_2u(b)&=C_2. \end{aligned} \tag{1.9}$$

where A_1, B_1 are not both zero and A_2, B_2 are not both zero. If $f(x)=0$ and $C_1=C_2=0,$ we say that the problem is homogenous boundary value problem. Otherwise, we say it is a non-homogenous boundary value problem. For instance,

the BVP $u''+u=0,$ with $u(0)=0$, $u(\pi)=0$ is homogenous and

the BVP $u''+u=1$, with $u(0)=0$, $u(2\pi)=0$ is non-homogenous.

Without any loss of generality, we can write the Equation (1.8) in the following form which will be discussed next.

$$(p(x)u')' + q(x)u(x) = g(x)\,. \qquad (1.10)$$

1.3.1 Sturm-Liouville Problem

As it is known that the dynamic problem for beam equation leads to an eigenvalue problem, the second-order dynamic problem would lead to an eigenvalue problem known is the Sturm-Liouville problem. It is of the form:

$$-(p(x)u')' + q(x)u = \lambda w(x)u\,, \qquad a < x < b\,, \qquad (1.11)$$

where $p(x), p'(x), q(x)$ and the weight function $w(x)$ are-real valued continuous functions on (a,b), $p(x) > 0$ and $w(x) > 0$, with the boundary contortions:

$$\begin{cases} \alpha_1 u(a) + \alpha_2 u'(a) = 0, \\ \beta_1 u(b) + \beta_2 u'(b) = 0, \end{cases} \qquad (1.12)$$

where α_1, α_2, β_1, β_2 are constants such that not both of α_1 and α_2 are zero and also not both β_1 and β_2 are zero. This means that the boundary conditions at least at the endpoints do not breakdown.

The values of λ for which these is non-trivial and the function $u(x)$ that satisfies the differential Equation (1.11) and the boundary conditions (1.12) is known as an eigenvalue of the Sturm-Liouville problem and the corresponding function $u(x)$ is the eigenfunction of the given problem.

We have two special cases of the boundary conditions:

i. $u(a)=0$, $u(b)=0$, (Dirichlet boundary conditions).

ii. $u'(a)=0$, $u'(b)=0$, (Neumann boundary conditions).

iii. $\begin{cases}\alpha_1 u(0)+\alpha_2 u(1)=0, \\ \beta_1 u'(0)+\beta_2 u'(1)=0.\end{cases}$ where $\alpha_1,\alpha_2,\beta_1,\beta_2 \in R$. (Robin boundary conditions).

Remark

The Sturm-Liouville problem (SLP) is said to be regular if the functions p,q and p' are continuous on the interval $[a,b]$, provided that $p' \neq 0$ on $[a,b]$ (in case we have a domain Ω as a general domain, it must be bounded). Otherwise, it is called singular Sturm-Liouville problem.

The interesting thing for a regular Sturm-Liouville problems is that have infinite number of eigenvalues and corresponding eigenfunctions that form a complete orthogonal set which leads us to the expansions of orthogonality. This is a good idea in applied mathematics and engineering.

Theorem 1.1

Let $p(x)$, $q(x)$ be real functions with $p(x)>0$ ($\forall x$ in the domain) in a regular Sturm-Liouville problem. Then the eigenvalues are real and the eigenfunctions are differentiable and continuous.

Theorem 1.2

The regular Sturm-Liouville problem has infinitely many real and simple eigenvalues λ_n, $n=0,1,2,...$ which can be arranged as $\lambda_0<\lambda_1<\lambda_2<...<\lambda_n<...$ with $\lambda_n \to \infty$, as

$n \to \infty$. *The eigenfunctions corresponding to the distinct eigenvalues are orthogonal and the set of all eigenfunctions is complete.*

1.4 Green's Function

Assume that a problem with data {0,0,0} has only the trivial solution. We want to prove that the problem:

$$\begin{cases} Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = f(x), a < x < b, \\ B_1u = \alpha_{11}u(a) + \alpha_{12}u'(a) + \beta_{11}u(b) + \beta_{12}u'(b) = \gamma_1, \\ B_2u = \alpha_{21}u(a) + \alpha_{22}u'(a) + \beta_{21}u(b) + \beta_{22}u'(b) = \gamma_2, \end{cases} \tag{1.13}$$

where $a_2(x)$, $a_1(x)$ and $a_0(x)$ are continuous functions on $[a,b]$, $a_2(x) \neq 0$, $\forall x \in [a,b]$ and $f(x)$ is piecewise continuous on $[a,b]$, with data $\{f, \gamma_1, \gamma_2\}$ has one and only one solution. This will be worked by using Green's function $G(x,\xi)$ which is a solution of the previous problem with data $\{\delta(x-\xi),0,0\}$, where ξ is fixed element on (a,b). Clearly, $\delta(x-\xi)$ is not a piecewise continuous function and we consider $G(x,\xi)$ to satisfy the boundary conditions $B_1G = 0$ and $B_2G = 0$.

The Green's function $G(x,\xi)$ for the second-order value problem is defined as:

$$LG(x,\xi) = \delta(x-\xi), \qquad a < x < \xi < b, \tag{1.14}$$

such as G is continuous at $x = \xi$ and:

$$\left.\frac{dG(x,\xi)}{dx}\right|_{x-\xi^+} - \left.\frac{dG(x,\xi)}{dx}\right|_{x-\xi^-} = \frac{1}{a_2(\xi)}.$$

Hence, Green's function is a solution of the previous homogenous problem. Problem 1.14 has at most one solution. If we assume that, there are two solutions $G_1(x,\xi)$ and $G_2(x,\xi)$ then, we must have:

$G_1(x,\xi) - G_2(x,\xi) = 0$ and so $G_1(x,\xi) = G_2(x,\xi)$. This means that Green's function is a unique solution of the Problem 1.14.

1.4.1 The Boundary Conditions

The boundary conditions that given in Problem 1.13 are unmixed if $\beta_{11}, \beta_{12}, \alpha_{21}, \alpha_{22} = 0$ and so we have:

$$\begin{aligned} B_1 u &= \alpha_{11} u(a) + \alpha_{12} u' a) = 0, \\ B_2 u &= \beta_{21} u(b) + \beta_{22} u'(b) = 0. \end{aligned}$$

The boundary conditions that given in Problem 1.13 reduces to an initial value problem if:

$$\alpha_{11} = \beta_{22} = 1,$$

and all $\alpha_{i,j}$, $\beta_{i,j}$ are zero (for $i \neq j$), i.e.

$$\begin{aligned} B_1 u &= u(a) = 0, \\ B_2 u &= u'(b) = 0. \end{aligned}$$

We describe below the method of finding Green's function for unmixed boundary conditions.

In order to find Green's function, we note that it satisfies the following problem:

$$a_2(x)G''(x,\xi) + a_1(x)G'(x,\xi) + a_0(x)G(x,\xi) = 0 \;,\; a < x < \xi < b \quad , \quad x \neq \xi \,, \tag{1.15}$$

with the boundary conditions:

$$\begin{aligned} B_1 u(a) &= \alpha_{11} G(a) + \alpha_{12} G'(a) + \beta_{11} G(b) + \beta_{12} G'(b) = 0, \\ B_2 u(b) &= \alpha_{11} G(a) + \alpha_{12} G'(a) + \beta_{11} G(b) + \beta_{12} G'(b) = 0. \end{aligned}$$

Let $u_1(x)$ be a linear solution of Equation (1.15) that satisfies B_1 and $u_2(x)$ be a linear independent solution that satisfies B_2 so that Green's function has the form:

$$G(x,\xi)=\begin{cases} Au_1(x), a<x<\xi, \\ Bu_2(x), \xi<x<b, \end{cases}$$

where A, B are constants.

The continuity of $G(x,\zeta)$ and jump condition of G' at $x=\xi$ give us:

$$\begin{aligned} &Au_1(\xi)-Bu_2(\xi)=0, \\ &-Au_1'(\xi)+Bu_2'(\xi)=\frac{1}{a_2(\xi)}. \end{aligned} \tag{1.16}$$

The inhomogeneous previous system (1.16) for A and B has one and only one solution if and only if $\begin{vmatrix} u_1(\xi) & -u_2(\xi) \\ -u_1'(\xi) & u_2'(\xi) \end{vmatrix} \neq 0$, which is the Wronskian of u_1 and u_2 evaluated at ξ. This Wronskian does not vanish anywhere, hence by solving Equations (1.16) we have the following:

$$A=\frac{u_2(\xi)}{a_2(\xi)W(u_1,u_2,\xi)} \quad , \quad B=\frac{u_1(\xi)}{a_2(\xi)W(u_1,u_2,\xi)}.$$

Thus, Green's function takes the form:

$$G(x,\xi)=\begin{cases} \dfrac{u_2(\xi)}{a_2(\xi)W(u_1,u_2,\xi)}u_1(x), & a<x<\xi, \\ \dfrac{u_1(\xi)}{a_2(\xi)W(u_1,u_2,\xi)}u_2(x), & \xi<x<b. \end{cases}$$

1.4.2 Initial Conditions

We can derive Green's function as:

$$G(x,\xi) = H(x-\xi)u_{\xi}(x),$$

where $u_{\xi}(x)$ is a solution of $Lu = 0$ that satisfies the conditions $u(\xi) = 0,\ u'(\xi) = \frac{1}{a_2(\xi)}$.

In case we have constant coefficients, the Green function can be written as the following new form:

$$G(x,\xi) = H(x-\xi)u_0(x-\xi),$$

where $H(x-\xi)$ is called Heaviside function (unit step function) and it is defined as following:

$$H(x-\xi) = \begin{cases} 0, & x < \xi, \\ 1, & \xi < x. \end{cases}$$

1.4.3 The General Boundary Conditions

Assume that $G(x,\xi)$ satisfies the mixed conditions $B_1G = B_2G = 0$, where, B_1 and B_2 are given in (1.13), then $G(x,\xi)$ can be written as following:

$$G(x,\xi) = H(x-\xi)u_{\xi}(x) + Au_1(x) + Bu_2(x),\quad a < x < \xi < b\ ,\quad x \neq \xi\ ,$$

where the first term is called the casual fundamental solution and both second and third terms are the solutions of the homogenous differential equation, u_1 and u_2 being the linearly independent solution of $Lu = 0$, with $B_1u = 0$ and $B_2u = 0$ respectively. We impose the boundary conditions to find the unknown:

$$\beta_{11}u_{\xi}(b)+\beta_{12}u'_{\xi}(b)+B(B_1u_2)=0,$$
$$\beta_{21}u_{\xi}(b)+\beta_{22}u'_{\xi}(b)+A(B_2u_1)=0.$$

Thus, we can solve for A and B.

Green's function constructed above for unmixed, initial and mixed conditions enables us to write the solution of homogenous boundary value problem:

$$Lu=f(x), \qquad a<x<b, \qquad (1.17)$$

with the boundary conditions:

$$B_1u=0,$$
$$B_2u=0.$$

In fact, the unique solution of (1.17) with the boundary conditions is given by:

$$u(x)=\int_a^b G(x,\xi)f(\xi)d\xi. \qquad (1.18)$$

If the boundary value problem has non-homogenous boundary conditions, i.e. with data $\{f,\gamma_1,\gamma_2\}$, we shall find Green's function $G(x,\xi)$ corresponding to homogenous boundary conditions. In such a case, the unique solution of:

$$Lu=f \ , \qquad a<x<b,$$

with the boundary conditions:

$$\begin{cases} B_1u=\gamma_{1,} \\ B_2u=\gamma_2, \end{cases}$$

is given by:

$$u(x)=\int_a^b G(x,\xi)\,f(\xi)d\xi+\frac{\gamma_2}{B_2u_1}u_1(x)+\frac{\gamma_1}{B_1u_2}u_2(x)\,, \tag{1.19}$$

where u_1 and u_2 are the non-trivial solutions of homogenous equation with $B_1u_1=0$ and $B_2u_2=0$, respectively.

Now, we can state the following theorem:

Theorem 1.3

If the completely homogenous boundary value problem has only the trivial solution, then the boundary value problem with data $\{f,\gamma_1,\gamma_2\}$ has one and only one solution given by Equation *(1.19).*

1.5 The Fourth-Order Differential Linear Beam Equation

As the Euler-Bernoulli beam equation is a fourth-order differential equation, it is useful to summarize some results about initial and boundary value problems of fourth-order.

Consider the following problem:

For all $a<x<b$, we have:

$$Lu=a_4(x)u^{(4)}(x)+a_3(x)u'''(x)+a_2(x)u''(x)+a_1(x)u'(x)+a_0(x)u(x)=f(x)\,, \tag{1.20}$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)\in C[a,b]$, $a_4(x)\neq 0,\forall x\in[a,b]$ and the function $f(x)$ is a piecewise continuous function on $[a,b]$.

If $u(x)$ is the general solution of the differential Equation (1.20), we impose the following boundary conditions:

$$
\begin{aligned}
B_1u &= \alpha_1 u(a) + \alpha_2 u'(a) + \alpha_3 u''(a) + \alpha_4 u'''(a) = \eta_1, \\
B_2u &= \beta_1 u(a) + \beta_2 u'(a) + \beta_3 u''(a) + \beta_4 u'''(a) = \eta_2, \\
B_3u &= \gamma_1 u(b) + \gamma_2 u'(b) + \gamma_3 u''(b) + \gamma_4 u'''(b) = \eta_3, \\
B_4u &= \xi_1 u(b) + \xi_2 u'(b) + \xi_3 u''(b) + \xi_4 u'''(b) = \eta_4.
\end{aligned}
\tag{1.21}
$$

The vectors $(\alpha_i),(\beta_i),(\gamma_i),(\eta_i)$ are linearly independent.

We point out that B_1, B_2, B_3, B_4 are boundary functional, because they specify sufficiently smooth function $u(x)$ the values B_1, B_2, B_3 and B_4 respectively.

The differential Equation (1.20) equipped with the boundary conditions (1.21) give us the boundary value problem (BVP).

Remarks

i. We assume that the coefficients $(\alpha_i),(\beta_i),(\gamma_i),(\xi_i)$ are known, then $\{f,\eta_1,\eta_2,\eta_3,\eta_4\}$ is the data of the boundary value problem (1.20,1.21).

ii. The independence of the vectors $(\alpha_1,\alpha_2,\alpha_3,\alpha_4),(\beta_1,\beta_2,\beta_3,\beta_4),$ $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ and $(\xi_1,\xi_2,\xi_3,\xi_4)$ guarantees that there are exactly four distinct boundary conditions.

iii. If the boundary value problem with the data $\{0,0,0,0,0\}$ has only a trivial solution $u \equiv 0$, then the boundary value problem with the data $\{f,\eta_1,\eta_2,\eta_3,\eta_4\}$ has at most one solution and if the boundary value problem has a non-trivial solution, then the boundary value problem with the data $\{f,\eta_1,\eta_2,\eta_3,\eta_4\}$ either has no solution or it has many solutions.

In the homogenous Euler-Bernoulli beam equation we have the fourth-order differential equation given by (see Eq. (1.7)):

$$EI\,\frac{d^4u}{dx^4} = f(x), \qquad (1.22)$$

where EI is constant. We thus focus on a boundary value problem involving a differential equation of the previous type.

In the following, we consider a boundary value problem with homogenous boundary conditions to describe Green's function associated with this equation.

Consider the following problem:

$$u^{(4)}(x) = f(x) \quad , \qquad 0 \le x \le a, \qquad (1.23)$$

subject to the conditions:

$$u(0) = u'(0) = u(a) = u'(a) = 0.$$

Eq. (1.23) can be written as:

$$D[u] = u^{(4)}(x) = f(x) \quad , \qquad 0 \le x \le a,$$

with the boundary conditions:

$$B_1u = u(0) = 0,\;\; B_4u = u^{'}(a) = 0,\;\; B_3u = u(a) = 0,\;\; B_2u = u^{'}(0) = 0.$$

Green's function $G(x,\xi)$ is defined as the solution of:

$$G_{xxxx}(x,\xi) = \delta(x-\xi), \qquad (1.24)$$

subjected to:

$$G(0,\xi)=G_{\mathrm{x}}(0,\xi)=0 \text{ and } G(a,\xi)=G_{\mathrm{x}}(\mathrm{a},\xi)=0. \qquad (1.25)$$

We assert that the Green function that satisfies above conditions gives the solution of the non-homogenous boundary value problem (1.22) as:

$$u(x)=\int_a^b G(x,\xi)f(\xi)d\xi .$$

In fact, assuming the integral above is uniformly convergent, we can write:

$$D\left[\int_0^a G(x,\xi)f(\xi)d\xi\right]=\int_0^a D[G(x,\xi)]f(\xi)d\xi=\int_0^a \delta(x-\xi)f(\xi)d\xi= f(\xi),$$

The solution $u(x)$ also satisfies the boundary conditions as:

$$B_i\left[\int_0^a G(x,\xi)f(\xi)d\xi\right]=\int_0^a B_i[G(x,\xi)]f(\xi)d\xi=\int_0^a (0)f(\xi)d\xi=0,\ i=1,2,3,4.$$

Now, our goal is to find the Green function $G(x,\xi)$. Green's function satisfying the following properties:

1. For each x and ξ, $G(x,\xi)$ satisfies $G_{xxxx}(x,\xi)=\delta(x-\xi)$, $x\neq\xi$.
2. $G(x,\xi)$ must satisfy the boundary conditions:

$$G(0,\xi)=G_x(0,\xi)=G(a,\xi)=G_x(a,\xi).$$

3. $G(x,\xi)$ and it's derivative up to 2^{nd} order are continuous at $x=\xi$, i.e.

$$G(\xi^+,\xi)-G(\xi^-,\xi)=0,$$

$$G_x(\xi^+,\xi)-G_x(\xi^-,\xi)=0,$$

$$\text{and } G_{xx}(\xi^+,\xi)-G_{xx}(\xi^-,\xi)=0.$$

4. The 3^{rd} order derivative of G has the jump discontinuous of magnitude 1 at $x=\xi$, i.e.

$$G_{xxx}(\xi^+,\xi)-G_{xxx}(\xi^-,\xi)=1.$$

According to the previous properties of the Green function, we can easily construct the second-order cases.

It is easy to see that the linearly independent solutions of the differential Equation (1.24) are:

$$1,x,x^2,x^3 \text{ and } 1,a-x,(a-x)^2,(a-x)^3.$$

We assume, to begin with, that $G(x,\xi)$ is given by:

$$G(x,\xi)=\begin{cases} a_1+a_2x+a_3x^2+a_4x^3, & 0\le x<\xi\le a, \\ b_1+b_2(a-x)+b_3(a-x)^2+b_4(a-x)^3, & 0\le \xi<x\le a, \end{cases} \tag{1.26}$$

where $a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4$ are functions of ξ.

As we know that the Green function $G(x,\xi)$ satisfies the homogenous boundary conditions (1.25), we must have:

$$a_1=0,\ a_2=0,\ b_1=0,\ b_2=0.$$

Hence, Green's function can be written as follows:

$$G(x,\xi)=\begin{cases} a_3x^2+a_4x^3, & 0\le x<\xi\le a, \\ b_3(a-x)^2+b_4(a-x)^3, & 0\le \xi<x\le a. \end{cases} \tag{1.27}$$

The continuity of $G(x,\xi)$, $G_x(x,\xi)$ and $G_{xx}(x,\xi)$ at $x=\xi$ implies:

$$\begin{cases} b_3(a-\xi)^2+b_4(a-\xi)^3=a_3\xi^2+a_4\xi^3, \\ -2b_3(a-\xi)-3b_4(a-\xi)^2=2a_3\xi+3a_4\xi^2, \\ 2b_3+6b_4(a-\xi)=2a_3+6a_4\xi. \end{cases}$$

The condition of the jump discontinuity of $G_{xxx}(x,\xi)$ at $x=\xi$ gives:

$$-6b_4 - 6a_4 = 1.$$

Using Mathematica program to find a_3, a_4, b_3, b_4, we get:

$$a_3 = \frac{\xi(a^2 - 2a\xi + \xi^2)}{2a^2},$$

$$a_4 = -\frac{2a\xi^3 - 3a\xi^2 + a^3}{6a^3},$$

$$b_3 = -\frac{\xi^2(\xi - a)}{2a^2},$$

$$b_4 = \frac{\xi^2(2\xi - 3a)}{6a^3}.$$

If we substitute these values into Equation (1.25), we have the following form:

$$G(x,\xi) = \begin{cases} \dfrac{x^2(\xi - a)^2(3a\xi - 2\xi x - ax)}{6a^3}, & 0 \le x < \xi \le a, \\ -\dfrac{\xi^2(a-x)^2(2a\xi + 2\xi x - 3ax)}{6a^3}, & 0 \le \xi < x \le a. \end{cases} \tag{1.28}$$

Note that the Green function $G(x,\xi)$ is symmetric, this means that $G(x,\xi) = G(x,\xi)$ and it's non-negative, i.e.

$$G_\xi(x,\xi) \ge 0, \quad \text{on} \quad 0 \le x \le \xi.$$

1.6 Spectral Theory of the Fourth-Order Differential Linear Operator $\frac{d^4}{dx^4}$

In this Section, we discuss special properties of the operator defined by the differential expression $\frac{d^4}{dx^4}$ in $H^4([0,1]) = W^{4,2}([0,1])$ with appropriate boundary conditions. These boundary conditions have been listed in Subsection 1.1.1.

1.6.1 Definitions (*Hilbert Space, Banach Space and Sobolev Space*)

1. *Hilbert Space*

Let H be a *vector space* over a field F. A functional $\langle \cdot,\cdot \rangle : H \times H \to F$ is said to be an inner (scalar) product on H, if the following conditions hold:

i. $\langle u,u \rangle \geq 0, \quad \forall u \in H$ (positive).

ii. $\langle u,u \rangle = 0,$ if and only if $u = 0$.

iii. $\langle u,v \rangle = \langle v,u \rangle, \quad \forall u,v \in H$ (symmetric).

iv. $\langle \alpha u + \beta v, w \rangle = \alpha \langle u,w \rangle + \beta \langle v,w \rangle, \quad \forall u,v,w \in H$ and for every $\alpha, \beta \in F$ (Linearity).

The pair $(H, \langle \cdot,\cdot \rangle)$ is called an inner (scalar) product space.

A *Hilbert space* is a vector space H equipped with a scalar product $\langle \cdot,\cdot \rangle$ such that H is complete for the norm $\|\cdot\|$ that induced by $\langle \cdot,\cdot \rangle$.

2. *Banach space*

Let X be a *vector space* over a field F. A functional $\|\cdot\| : X \to F$ is said to be a norm on X, if the following conditions hold:

i. $\|u\| \geq 0, \quad \forall u \in X$ (positive).

ii. $\|u\| = 0,$ if and only if $u = 0$.

iii. $\|\alpha u\| = |\alpha| \|u\|, \quad \forall u \in X$, for each $\alpha \in \mathrm{F}$.

iv. $\|u + v\| \leq \|u\| + \|v\|, \quad \forall u,v \in X$.

The pair $(X, \|\cdot\|)$ is called a norm space.

A *Banach space* is a vector space X equipped with a normed space $\|\cdot\|$ such that X is complete for the norm $\|\cdot\|$.

3. ***Sobolev Space***

For any given integer $m \geq 2$, we define the *Sobolev space* of order m by the following:

$$W^{m,p}(I) = \{u \in L^p(I) : u', u'', \ldots, u^{(m)} \in L^p(I), \text{ where } p = 1, 2, \ldots\}.$$

The derivatives are in the weak sense. Then we can write the following:

$$H^m(I) = W^{m,2}(I).$$

1.6.2 Adjoint of the Fourth-Order Differential Linear Operator $T \cong \frac{d^4}{dx^4}$

Define an operator $T : D(T) \to L^2[0,1]$,

where:
$D(T) = \{u \in L^2[0,1] : u', u'', u''', u^{(4)} \in L^2[0,1] \text{ and satisfying the boundary conditions}\}.$

We see that, use of integration by parts leads us to have:

$$\langle Tu, v\rangle = \int_0^1 Tuv\,dx = [u,v] + \int_0^1 u \frac{d^4 v}{dx^4}\,dx = [u,v] + \langle u, Tv\rangle,$$

where the bilinear form $[u,v]$ is given by:

$$[u,v] = u'''v - u''v' + u'v'' - uv'''\Big|_0^1.$$

Clearly, the bilinear form $[u,v]$ depends in which type upon the boundary conditions are imposed.

In this case, let us discuss the form of the bilinear form $[u,v]$ regarding to the three types of boundary conditions (hinged boundary conditions, clamped boundary conditions and clamped-free boundary conditions).

i. **The Hinged Boundary Conditions**

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

in this case, we have:

$$[u,v] = u'''v\Big|_0^1 + u'v''\Big|_0^1.$$

We see that if we impose the same boundary conditions on v i.e.

$$v(0) = v(1) = v''(0) = v''(1) = 0,$$

we have:

$$\langle Tu, v\rangle = \langle u, Tv\rangle.$$

This means that the adjoin operator of T is again T itself with the same boundary conditions by T.

Thus $T \cong \dfrac{d^4}{dx^4}$, with $D(T)$ consisting of $u(x)$ such that $u'(x), u''(x), u'''(x)$ and $u^{(4)}(x)$ are in $L^2([0,1])$ for all $x \in [0,1]$ satisfying hinged boundary conditions is a self-adjoin operator.

ii. **The Clamped Boundary Conditions**

$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

We again consider the bilinear form:

$$[u,v] = u'''v\big|_0^1 - u''v'\big|_0^1 .$$

We see that if we impose the same boundary conditions on v, i.e.

$$v(0) = v(1) = v'(0) = v'(1) = 0,$$

we have:

$$\langle Tu, v\rangle = \langle u, Tv\rangle .$$

This means that the adjoin operator of T is again T itself with the same boundary conditions by T.

Thus $T \cong \dfrac{d^4}{dx^4}$, with $D(T)$ consisting of $u(x)$ such that $u'(x), u''(x), u'''(x)$ and $u^{(4)}(x)$ are in $L^2([0,1])$ for all $x \in [0,1]$ satisfying clamped boundary conditions is a self-adjoin operator.

iii. One End Clamped-One End Free

$$u(0) = u'(0) = u''(1) = u'''(1) = 0 .$$

The bilinear form takes the form:

$$[u,v] = -u'''v\big|_{x=0} + u''v'\big|_{x=0} + u'v''\big|_{x=1} uv'''\big|_{x=1} = 0 .$$

We see that if we impose the same boundary conditions on v, i.e.

$$v(0) = v'(0) = v''(1) = v'''(1) = 0,$$

we have:

$$\langle Tu, v\rangle = \langle u, Tv\rangle .$$

This means that the adjoin operator of T is again T itself with the same boundary conditions by T.

Thus $T \cong \frac{d^4}{dx^4}$, with $D(T)$ consisting of $u(x)$ such that $u'(x), u''(x), u'''(x)$ and $u^{(4)}(x)$ are in $L^2([0,1])$ for all $x \in [0,1]$ satisfying one end clamped-one free boundary conditions is a self-adjoin operator.

We have shown that, the fourth-order differential expression $T \cong \frac{d^4}{dx^4}$, with appropriate boundary conditions Hinged-Hinged, Clamped-Clamped and Clamped-Free define self-adjoin operators. We also have $\langle Tu, u \rangle \geq 0$, i.e. T is positive operator.

For,

$$\langle Tu, u \rangle = \langle \frac{d^4 u}{dx^4}, u \rangle = \int_0^1 \frac{d^4 u}{dx^4} u dx = u \frac{d^3 u}{dx^3}\bigg|_0^1 - \int_0^1 \frac{d^3 u}{dx^3} u dx$$

$$= u \frac{d^3 u}{dx^3}\bigg|_0^1 - \frac{du}{dx}\frac{d^2 u}{dx^2}\bigg|_0^1 + \int_0^1 \frac{d^2 u}{dx^2}\frac{d^2 u}{dx^2} dx,$$

or as simply:

$$\langle Tu, u \rangle = uu'''\big|_0^1 - u'u''\big|_0^1 + \int_0^1 (u'')^2 dx .$$

In each case of boundary conditions above, we can see that the boundary terms vanish and thus we have:

$$\langle Tu, u \rangle = \int_0^1 (u'')^2 dx \geq 0 \quad \Rightarrow \quad \langle Tu, u \rangle \geq 0 .$$

Thus, T is a positive operator. Consequently, we have:

i. The eigenvalues of $T \cong \frac{d^4}{dx^4}$ (in each case of the above boundary conditions) are real, positive and simple.

ii. The eigenfunctions corresponding to the distinct eigenvalues form an orthogonal set, we have, $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n, \ldots$, where $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$, $n = 1,2,3,\ldots$ and eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots \phi_n, \ldots$, $n = 1,2,3,\ldots$, where $\{\phi_i\}\big|_{i=1}^{\infty}$ is an orthogonal set. i.e.

The inner (scalar) product of any two distinct eigenfunctions is zero, i.e.

$\langle u, v \rangle = 0,$ for all u and v belong to the domain of the operator.

Any function $f \in H$ can be written as eigenfunctions expansion by:

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \phi_i ,$$

where:

$$\alpha_i = \frac{\langle f(x), \phi_i(x) \rangle}{\langle \phi_i(x), \phi_i(x) \rangle}, \qquad i = 1,2,3,\ldots \ .$$

CHAPTER 2

GREEN'S FUNCTION FOR FOURTH-ORDER DIFFERENTIAL LINEAR OPERATOR

We have seen that the Euler-Bernoulli beam equation can be written as a fourth-order differential equation given by:

$$\frac{d^4u}{dx^4} = f(x), \qquad 0 \le x \le 1. \tag{2.1}$$

Together with appropriate boundary conditions arising from the manner how the beam is being held. In this chapter, we apply the procedure outlined in Chapter 1 to construct a Green's function.

2.1 Construction of Green's Function

1) Hinged Ends Case

The Green function corresponding to the non-homogeneous differential equation satisfies the following equation for Euler-Bernoulli beam hinged at both ends:

$$\frac{d^4G(x,\xi)}{dx^4} = \delta(x-\xi), \qquad 0 \le x \le 1, \quad x \ne \xi\,, \tag{2.2}$$

$$G(0,\xi) = 0,\ G_{xx}(0,\xi) = 0, \tag{2.2.i}$$

$$G(1,\xi) = 0,\ G_{xx}(1,\xi) = 0. \tag{2.2.ii}$$

In addition, $G(x,\xi)$ satisfies the continuity conditions at $x=\xi$, so we have:

$$\begin{cases} G(\xi^{+},\xi)-G(\xi^{-},\xi)=0, \\ G_{x}(\xi^{+},\xi)-G_{x}(\xi^{-},\xi)=0, \\ G_{xx}(\xi^{+},\xi)-G_{xx}(\xi^{-},\xi)=0. \end{cases} \tag{2.2.iii}$$

Also, it satisfies the jump discontinuity condition at $x=\xi$, so we have:

$$G_{xxx}(\xi^{+},\xi)-G_{xxx}(\xi^{-},\xi)=1. \tag{2.2.iv}$$

The linearly independent solutions of the differential Equation (2.2) are given by:

$$1, x, x^2, x^3 \text{ and } 1, 1-x, (1-x)^2, (1-x)^3.$$

As outlined in Chapter 1, we start with following form of Green's function:

$$G(x,\xi)=\begin{cases} A_0+A_1x+A_2x^2+A_3x^3, & 0\le x<\xi\le 1, \\ B_0+B_1(1-x)+B_2(1-x)^2+B_3(1-x)^3, & 0\le \xi<x\le 1, \end{cases} \tag{2.3}$$

where A_0, A_1, A_2, A_3 and B_0, B_1, B_2, B_3 are functions on ξ.

We now impose the boundary conditions (2.2.i) at $x=0$ and (2.2.ii) at $x=1$, we find:

$$G(0,\xi)=0 \text{ gives } A_0=0,$$

$$G_{xx}(0,\xi)=0 \text{ gives } A_2=0.$$

Similarly,

$$G(1,\xi)=0 \text{ gives } B_0=0,$$

$$G_{xx}(1,\xi)=0 \text{ gives } B_2=0.$$

Thus,

$$G(x,\xi)=\begin{cases} A_1x+A_3x^3, & 0\le x<\xi\le 1, \\ B_1(1-x)+B_3(1-x)^3, & 0\le \xi<x\le 1. \end{cases} \tag{2.4}$$

Now, we use the continuity conditions that satisfied by Green's function at $x=\xi$, to get:

$$B_1(1-\xi)+B_3(1-\xi)^3=A_1\xi+A_3\xi^3, \tag{2.5}$$

$$-B_1-3B_3(1-\xi)^2=A_1+3A_3\xi^2, \tag{2.6}$$

$$A_3=\frac{1-\xi}{\xi}B_3. \tag{2.7}$$

Again, we use the jump discontinuity condition that satisfied by Green's function at $x=\xi,$ we have:

$$B_3=\frac{-1-6A_3}{6}. \tag{2.8}$$

Put Eq. (2.8) in Eq. (2.7), we get:

$$A_3=\frac{1}{6}(\xi-1). \tag{2.9}$$

Put Eq. (2.9) in Eq. (2.8), we have:

$$B_3=\frac{-1}{6}\xi. \tag{2.10}$$

Substitute Eq. (2.10) and Eq. (2.9) in Eq. (2.6), to get:

$$B_1=\frac{1}{2}\xi(1-\xi)-A_1. \tag{2.11}$$

Now, put Eq. (2.9), Eq. (2.10) and Eq. (2.11) in Eq. (2.5), we have:

$$A_1=\frac{-1}{6}\xi(\xi-1). \tag{2.12}$$

Put Eq. (2.12) in Eq. (2.11), we get:

$$B_1=\frac{1}{3}\xi(1-\xi). \tag{2.13}$$

Now, substitute Eq. (2.13), Eq. (2.12), Eq. (2.10) and Eq. (2.9) in Eq. (2.4), we have:

$$G(x,\xi)=\begin{cases}\frac{1}{6}\xi(1-\xi)x+\frac{1}{6}(\xi-1)x^3, & 0\le x<\xi\le 1,\\ \frac{1}{3}\xi(1-\xi)(1-x)+\frac{1}{6}\xi(x-1)^3, & 0\le \xi<x\le 1.\end{cases} \tag{2.14}$$

Which is the Green function for the hinged ends case.

Remark

We note that all the conditions are satisfied by Green's function given by Eq. (2.14). We also note that $G(x,\xi)=G(\xi,x)$, i.e. the Green function is symmetric as expected.

2) **Clamped Ends Case**

The Green function corresponding to inhomogeneous differential equation satisfies the following equation for Euler-Bernoulli beam clamped at both ends:

$$\frac{d^4G(x,\xi)}{dx^4}=\delta(x-\xi), \qquad 0\le x\le 1, \quad x\ne\xi\,, \tag{2.15}$$

$$G(0,\xi)=0,\ G_x(0,\xi)=0\,, \tag{2.15.i}$$

$$G(1,\xi)=0,\ G_x(1,\xi)=0\,. \tag{2.15.ii}$$

In addition, $G(x,\xi)$ satisfies the continuity conditions at $x=\xi$, so we have:

$$\begin{cases}G(\xi^+,\xi)-G(\xi^-,\xi)=0,\\ G_x(\xi^+,\xi)-G_x(\xi^-,\xi)=0,\\ G_{xx}(\xi^+,\xi)-G_{xx}(\xi^-,\xi)=0.\end{cases} \tag{2.15.iii}$$

Also, it satisfies the jump discontinuity condition at $x=\xi$, so we have:

$$G_{xxx}(\xi^+,\xi)-G_{xxx}(\xi^-,\xi)=1\,. \tag{2.15.iv}$$

The linearly independent solutions of the differential Equation (2.15) are given by:

$$1, x, x^2, x^3 \text{ and } 1, 1-x, (1-x)^2, (1-x)^3.$$

As outlined in Chapter 1, we start with following form of Green's function:

$$G(x,\xi) = \begin{cases} A_0 + A_1 x + A_2 x^2 + A_3 x^3, & 0 \le x < \xi \le 1, \\ B_0 + B_1(1-x) + B_2(1-x)^2 + B_3(1-x)^3, & 0 \le \xi < x \le 1, \end{cases} \tag{2.16}$$

where A_0, A_1, A_2, A_3 and B_0, B_1, B_2, B_3 are functions on ξ.

We now impose the boundary conditions (2.15.i) at $x = 0$ and (2.15.ii) at $x = 1$, we find:

$$G(0,\xi) = 0 \text{ gives } A_0 = 0,$$

$$G_x(0,\xi) = 0 \text{ gives } A_1 = 0.$$

Similarly,

$$G(1,\xi) = 0 \text{ gives } B_0 = 0,$$

$$G_x(1,\xi) = 0 \text{ gives } B_1 = 0.$$

Thus,

$$G(x,\xi) = \begin{cases} A_2 x^2 + A_3 x^3, & 0 \le x < \xi \le 1, \\ B_2(1-x)^2 + B_3(1-x)^3, & 0 \le \xi < x \le 1. \end{cases} \tag{2.17}$$

Now, we use the continuity conditions that satisfied by Green's function at $x = \xi$, to get:

$$B_2(1-\xi)^2 + B_3(1-\xi)^3 = A_2\xi^2 + A_3\xi^3, \tag{2.18}$$

$$-2B_2(1-\xi) - 3B_3(1-\xi)^2 = 2A_2\xi + 3A_3\xi^2, \tag{2.19}$$

$$2B_2 + 6B_3(1-\xi) = 2A_2 + 6a_3\xi. \tag{2.20}$$

Again, using the jump discontinuity condition that satisfied by Green's function at $x = \xi$, we get:

$$B_3 = -(\frac{1}{6} + A_3)\,. \tag{2.21}$$

Put Eq. (2.21) in Eq. (2.20), we get:

$$B_2 = A_2 + 3A_3 + \frac{1}{2}(1-\xi)\,. \tag{2.22}$$

Put Eq. (2.22) and Eq. (2.21) in Eq. (2.19), we have:

$$A_2 = \frac{1}{4}(-6A_3 - 1 + 2\xi - \xi^2)\,. \tag{2.23}$$

Substitute Eq. (2.23), Eq. (2.22) and Eq. (2.21) in Eq. (2.18), to get:

$$A_3 = \frac{1}{6}(3\xi^2 - 2\xi^3 - 1)\,. \tag{2.24}$$

Now, put Eq. (2.24) in Eq. (2.23), we have:

$$A_2 = \frac{1}{2}\xi(\xi^2 - 2\xi + 1)\,. \tag{2.25}$$

Put Eq. (2.25) and Eq. (2.24) in Eq. (2.22), we get:

$$B_2 = \frac{1}{2}\xi^2(1-\xi)\,. \tag{2.26}$$

Put Eq. (2.26) in Eq. (2.21), we get:

$$B_3 = \frac{1}{6}\xi^2(2\xi - 3)\,. \tag{2.27}$$

Now, substitute Eq. (2.27), Eq. (2.26), Eq. (2.25) and Eq. (2.24) in Eq. (2.17), we have:

$$G(x,\xi) = \begin{cases} \frac{1}{2}\xi(\xi^2 - 2\xi + 1)x^2 + \frac{1}{6}(3\xi^2 - 2\xi^3 - 1)x^3, & 0 \le x < \xi \le 1, \\ \frac{1}{2}\xi^2(1-\xi)(1-x)^2 + \frac{1}{6}\xi^2(2\xi - 3)(x-1)^3, & 0 \le \xi < x \le 1. \end{cases} \tag{2.28}$$

Which is the Green function for the clamped ends case.

Remark

We note that all the conditions are satisfied by Green's function given by Eq. (2.28). We also note that $G(x,\xi)=G(\xi,x)$, i.e. the Green function is symmetric as expected.

3) **One End Clamped-One Free**

The Green function corresponding to inhomogeneous differential equation satisfies the following equation for Euler-Bernoulli beam clamped at one end, one free:

$$\frac{d^4G(x,\xi)}{dx^4}=\delta(x-\xi), \qquad 0\le x\le 1, \quad x\ne\xi, \tag{2.29}$$

$$G(0,\xi)=0,\ G_x(0,\xi)=0, \qquad \text{(clamped end)} \tag{2.29.i}$$

$$G_{xx}(1,\xi)=0,\ G_{xxx}(1,\xi)=0, \qquad \text{(free end)} \tag{2.29.ii}$$

In addition, $G(x,\xi)$ satisfies the continuity conditions at $x=\xi$, so we have:

$$\begin{cases} G(\xi^+,\xi)-G(\xi^-,\xi)=0, \\ G_x(\xi^+,\xi)-G_x(\xi^-,\xi)=0, \\ G_{xx}(\xi^+,\xi)-G_{xx}(\xi^-,\xi)=0. \end{cases} \tag{2.29.iii}$$

Also, it satisfies the jump discontinuity condition at $x=\xi$, so we have:

$$G_{xxx}(\xi^+,\xi)-G_{xxx}(\xi^-,\xi)=1. \tag{2.29.iv}$$

The linearly independent solutions of the differential Equation (2.29) are given by:

$$1, x, x^2, x^3 \text{ and } 1, 1-x, (1-x)^2, (1-x)^3.$$

As outlined in Chapter 1, we start with following form of Green's function:

$$G(x,\xi)=\begin{cases}A_0+A_1x+A_2x^2+A_3x^3, & 0\le x<\xi\le 1,\\ B_0+B_1(1-x)+B_2(1-x)^2+B_3(1-x)^3, & 0\le \xi<x\le 1,\end{cases} \tag{2.30}$$

where A_0, A_1, A_2, A_3 and B_0, B_1, B_2, B_3 are functions on ξ.

We now impose the boundary conditions (2.29.i) at $x=0$ and (2.29.ii) at $x=1$, we find:

$$G(0,\xi)=0 \text{ gives } A_0=0,$$

$$G_x(0,\xi)=0 \text{ gives } A_1=0.$$

Similarly,

$$G_{xx}(0,\xi)=0 \text{ gives } B_2=0,$$

$$G_{xxx}(0,\xi)=0 \text{ gives } B_3=0.$$

Thus,

$$G(x,\xi)=\begin{cases}A_2x^2+A_3x^3, & 0\le x<\xi\le 1,\\ B_0+B_1(1-x), & 0\le \xi<x\le 1.\end{cases} \tag{2.31}$$

Now, we use the continuity conditions that satisfied by Green's function at $x=\xi$, to get:

$$B_0+B_1(1-\xi)=A_2\xi^2+A_3\xi^3, \tag{2.32}$$

$$-B_1=2A_2\xi+3A_3\xi^2, \tag{2.33}$$

$$2A_2+6A_3\xi=0\ . \tag{2.34}$$

Again, we use the jump discontinuity condition that satisfied by Green's function at $x=\xi$, we have:

$$A_3=\frac{-1}{6}\ . \tag{2.35}$$

Put Eq. (2.35) in Eq. (2.34), we get:

$$A_2 = \frac{1}{2}\xi \,. \qquad (2.36)$$

Put Eq. (2.36) and Eq. (2.35) in Eq. (2.33), we have:

$$B_1 = \frac{-1}{2}\xi^2 \,. \qquad (2.37)$$

Substitute Eq. (2.37), Eq. (2.36) and Eq. (2.35) in Eq. (2.32), to get:

$$B_0 = \frac{1}{6}\xi^2(3-\xi) \,. \qquad (2.38)$$

Now, substitute Eq. (2.38), Eq. (2.37), Eq. (2.36) and Eq. (2.35) in Eq. (2.31), we have:

$$G(x,\xi) = \begin{cases} \frac{1}{2}\xi x^2 - \frac{1}{6}x^3, & 0 \le x < \xi \le 1, \\ \frac{1}{6}\xi^2(3-\xi) - \frac{1}{2}\xi^2(1-x), & 0 \le \xi < x \le 1. \end{cases} \qquad (2.39)$$

Which is the Green function for the clamped at one end, one free case.

Remark

Note that all the conditions are satisfied by Green's function given by Eq. (2.39). Also, $G(x,\xi) = G(\xi,x)$, i.e. the Green function is symmetric as expected.

2.1.1 Solutions of Non-Homogenous Problem

In this Section, we get the general solutions of our previous problems.

a) Hinged Both Ends

We have seen that in Section 2.1, Green’s function $G(x,\xi)$ has been constructed by Eq. (2.14), so that the solution of beam equation with both ends satisfying hinged boundary conditions is given by:

$$u(x) = \int_0^1 G(x,\xi)\, f(\xi)\, d\xi$$

$$= \int_0^x \left[\frac{1}{3}\xi(1-\xi)(1-x) + \frac{1}{6}\xi(x-1)^3 \right] f(\xi)\xi$$
$$+ \int_x^1 \left[\frac{1}{6}\xi(1-\xi)x + \frac{1}{6}(\xi-1)x^3 \right] f(\xi)d\xi. \tag{2.40}$$

For instance, take $f(\xi) = \xi^2$, the solution will be:

$$u(x) = \frac{1}{360} x\left(3 - 5x^2 + 15x^3 - 24x^4 + 11x^5\right). \tag{2.41}$$

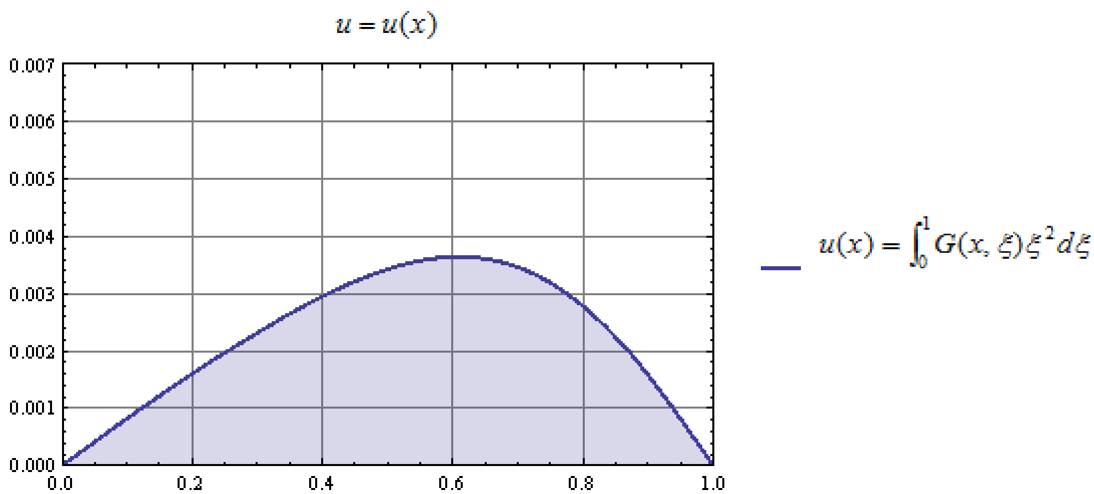

Figure 2.1: ***Solution of hinged ends, when*** $f(\xi) = \xi^2$

Also, when $f(\xi) = \sin(\xi)$, the solution will be:

$$u(x) = \frac{1}{6}\Big[4 - 4x - 2x\cos(1) - (4 - 5x - 2x^2 + x^3)\cos(x)$$
$$- x\sin(x) + x^3\sin(1) + (1 - 4x + 3x^2)\Big]. \tag{2.42}$$

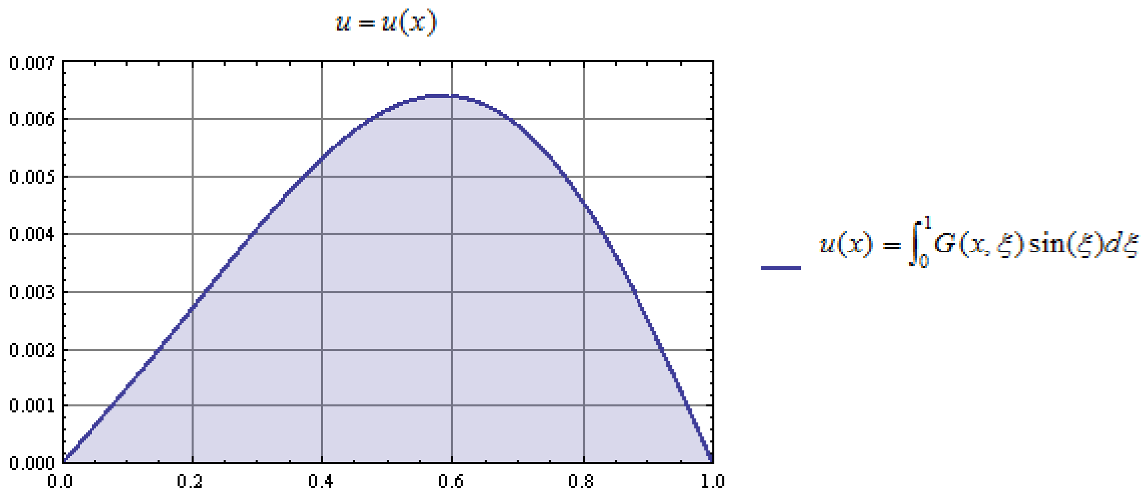

Figure 2.2: ***Solution of hinged ends, when*** $f(\xi) = \sin(\xi)$

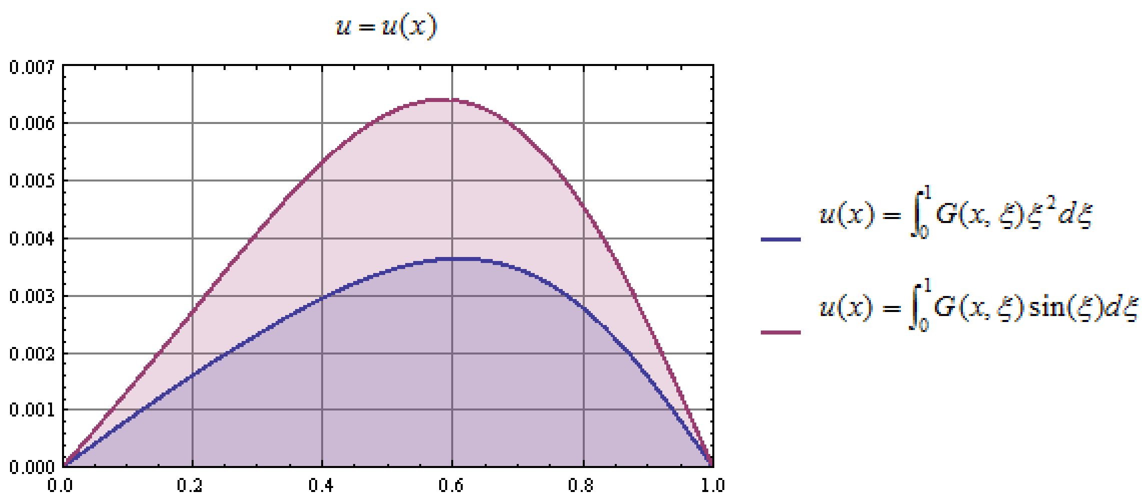

Figure 2.3: ***Gathering solutions of hinged ends of Fig. 2.1 and Fig. 2.2***

b) Clamped Both Ends

We have seen that in Section 2.1, Green's function $G(x,\xi)$ has been constructed by Eq. (2.28), so that the solution of beam equation with both ends satisfying clamped boundary conditions is given by:

$$u(x)=\int_0^1 G(x,\xi)f(\xi)d\xi$$

$$=\int_0^x\left[\frac{1}{2}\xi^2(1-\xi)(1-x)^2+\frac{1}{6}\xi^2(2\xi-3)(x-1)^3\right]f(\xi)d\xi$$
$$+\int_x^1\left[\frac{1}{2}\xi(\xi^2-2\xi+1)x^2+\frac{1}{6}(3\xi^2-2\xi^3-1)x^3\right]f(\xi)d\xi. \tag{2.43}$$

For instance, take $f(\xi)=\xi^2$, the solution will be:

$$u(x)=\frac{1}{360}(-1+x)^2x^2\left(3+2x+x^2 72x^3-112x^4+40x^5\right). \tag{2.44}$$

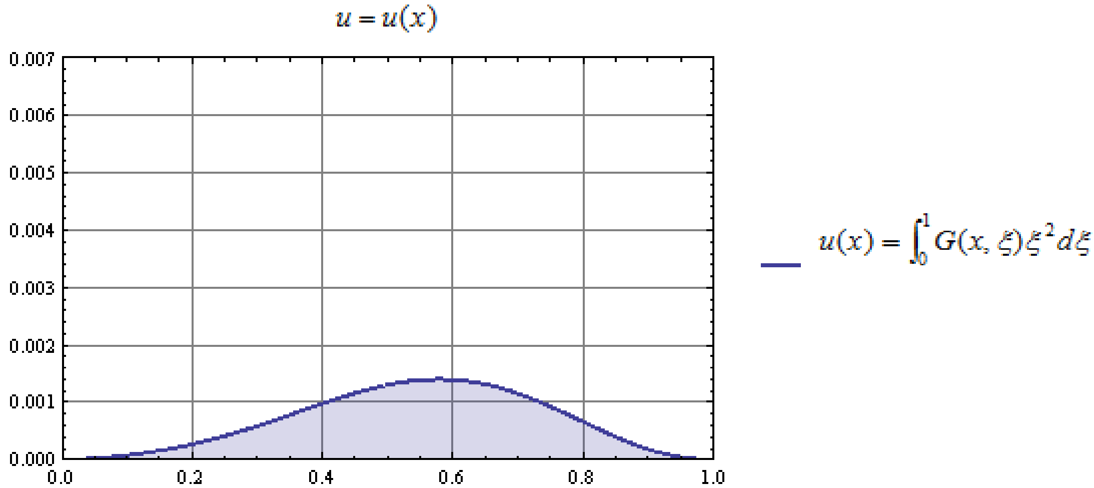

Figure 2.4: ***Solution of clamped ends, when*** $f(\xi)=\xi^2$

Also, when $f(\xi)=\sin(\xi)$, the solution will be:

$$u(x)=\frac{1}{3}\Big[-(-1+x)^3\left(6-12x-3x^2+2x^3\right)\cos(x)+3[-2+5x$$
$$+x^2\left(-4+\cos(1)-3\sin(1)\right)x^3\left(1-\cos(1)+2\sin(1)\right)$$
$$+\left(5-10x+4x^2+8x^3-8x^4+2x^5\right)\sin(x)]\Big]. \tag{2.45}$$

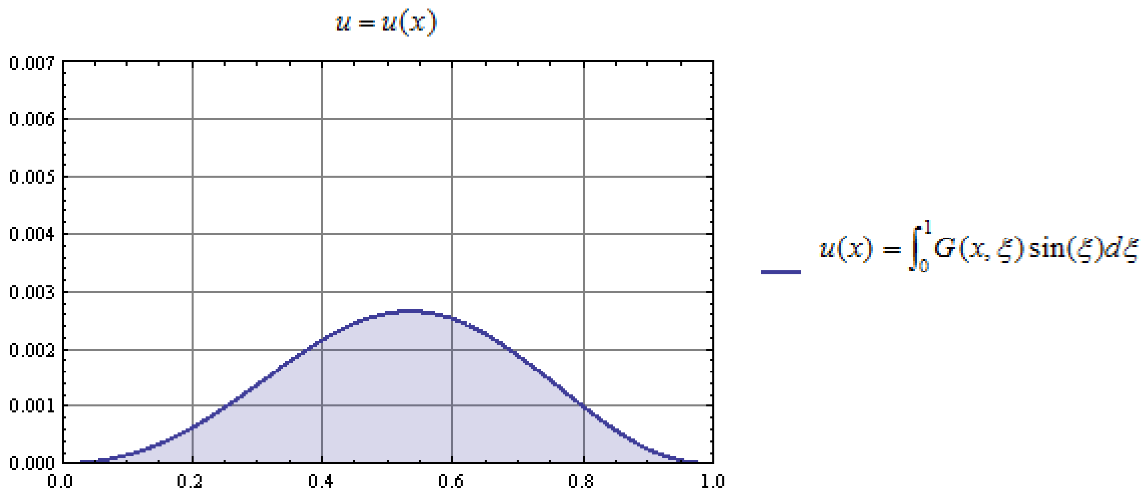

Figure 2.5: ***Solution of clamped ends, when*** $f(\xi) = \sin(\xi)$

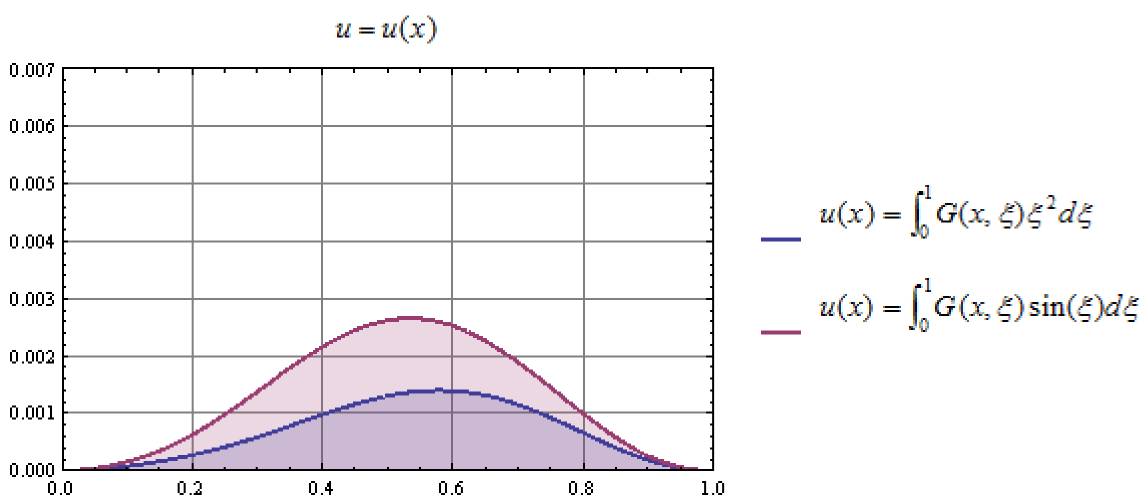

Figure 2.6: ***Gathering Solutions of clamped ends of Fig. 2.4 and Fig. 2.5***

c) Clamped at One End-One Free

We have seen that in Section 2.1, Green's function $G(x,\xi)$ has been constructed by Eq. (2.39), so that the solution of beam equation with both ends satisfying hinged boundary conditions is given by:

$$u(x) = \int_0^1 G(x,\xi) f(\xi) d\xi$$

$$= \int_0^x \left[\frac{1}{6}\xi^2(3-\xi) - \frac{1}{2}\xi^2(1-x) \right] f(\xi)d\xi + \int_x^1 \left[\frac{1}{2}\xi x^2 - \frac{1}{6}x^3 \right] f(\xi)d\xi \,. \tag{2.46}$$

For instance, take $f(\xi) = \xi^2$, the solution will be:

$$u(x) = \frac{1}{360}x^2\left(45 - 20x + x^4\right). \tag{2.47}$$

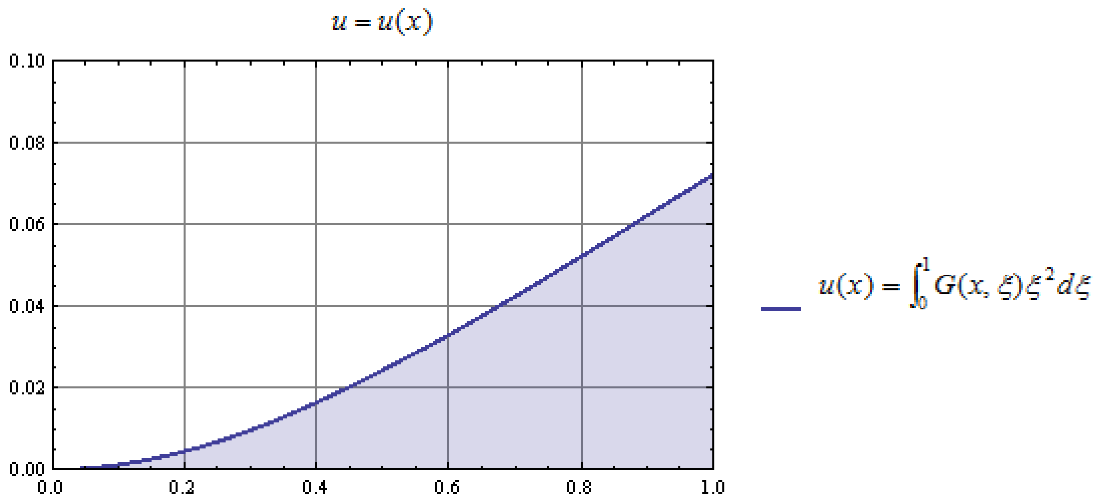

Figure 2.7: ***Solution of clamped at one-one free, when*** $f(\xi) = \xi^2$

Also, when $f(\xi) = \sin(\xi)$, the solution will be:

$$u(x) = \frac{1}{6}\left[x\left(-6 - 3x\cos(1) + x^2\cos(1) + 3x\sin(1) + 6\sin(x)\right)\right]. \tag{2.48}$$

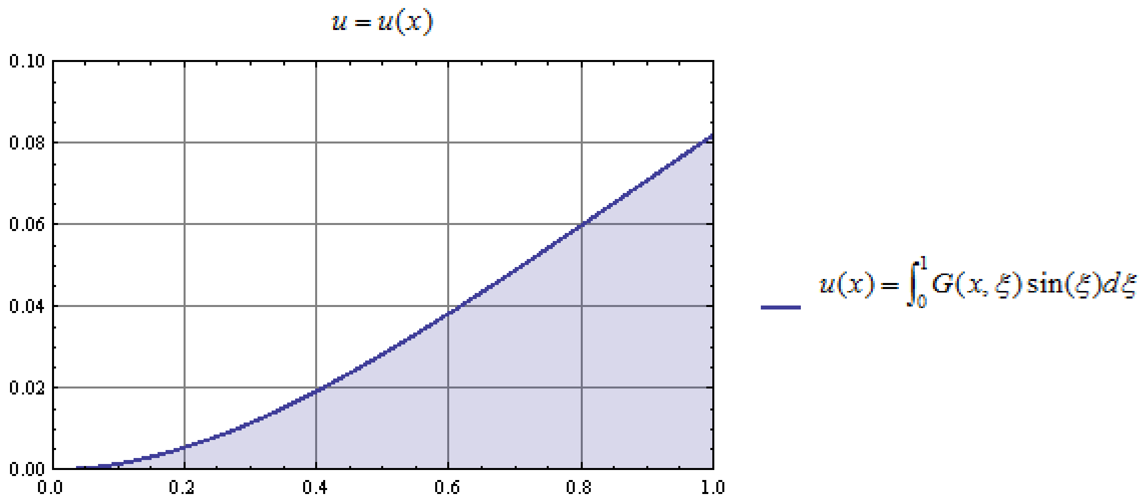

Figure 2.8: ***Solution of clamped at one-one free, when*** $f(\xi) = \sin(\xi)$

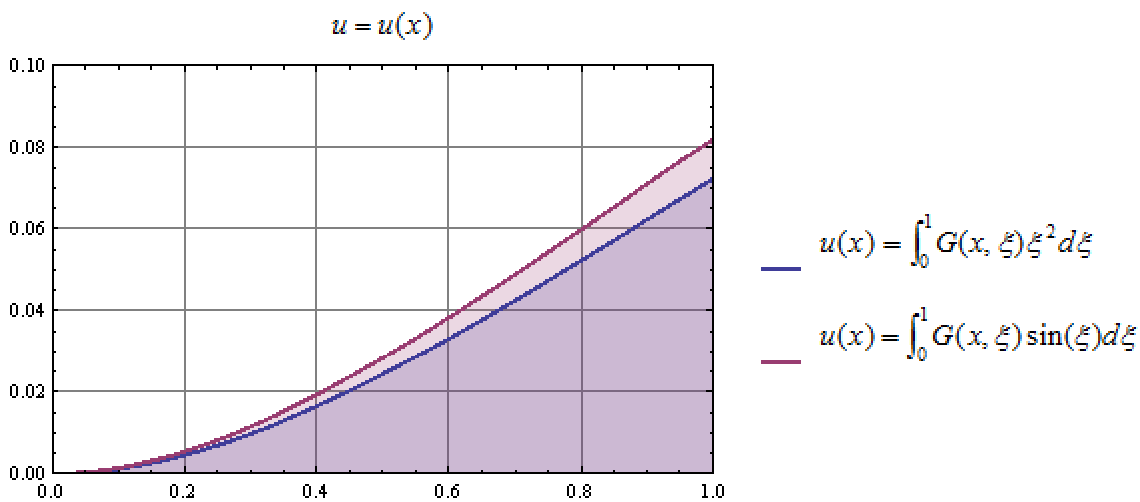

Figure 2.9: ***Gathering solutions of clamped at one-one free of Figs. 2.7 and Fig. 2.8***

2.1.2 Solutions of Completely Non-Homogenous Problem

Let us assume that we have the following problrm:

$$\frac{d^4u}{dx^4} = f(x), \qquad 0 \le x \le 1, \tag{2.49}$$

together with inhomogeneous boundary conditions as:

$$B_1 u = \alpha_1,\, B_2 u = \alpha_2, \qquad (2.49\text{i})$$
$$B_1 u = \beta_1,\, B_2 u = \beta_2.$$

Note that the solution $u(x)$ of the above problem (2.49) with boundary conditions given in (2.49i) can be written as following:

$$u(x) = v(x) + w(x), \qquad (2.50)$$

where $v(x)$ satisfies the following:

$$\frac{d^4 v(x)}{dx^4} = f(x), \qquad 0 \le x \le 1, \qquad (2.51)$$

with homogenous boundary conditions given as:

$$B_1 v = 0,\, B_2 v = 0, \qquad (2.51\text{i})$$
$$B_1 v = 0,\, B_2 v = 0,$$

and $w(x)$ satisfies the following:

$$\frac{d^4 w(x)}{dx^4} = 0, \qquad 0 \le x \le 1, \qquad (2.52)$$

with inhomogeneous boundary conditions given as:

$$B_1 w = \alpha_1,\, B_2 w = \alpha_2, \qquad (2.52\text{i})$$
$$B_1 w = \beta_1,\, B_2 w = \beta_2.$$

Now, as we have shown above in Section 2.1, the solution of problem (2.51) with homogenous boundary conditions (2.51i) is:

$$v(x) = \int_0^1 G(x,\xi) f(\xi) d\xi \,. \qquad (2.53)$$

The solution of problem (2.52) with inhomogeneous boundary conditions (2.52i) is:

$$w(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3. \qquad (2.54)$$

Remark

We can rewrite Equation (2.54) as the general solution of (2.52) with (2.52i) as:

$$w(x) = \frac{\alpha_2}{B_2 v_1} v_1(x) + \frac{\alpha_1}{B_1 v_2} v_2(x) + \frac{\alpha_4}{B_4 v_3} v_3(x) + \frac{\alpha_3}{B_3 v_4} v_4(x)\,, \qquad (2.54)$$

where $v_1(x), v_2(x), v_3(x), v_4(x)$ are non-trivial solutions of the homogenous equation that satisfying $B_1 v_1 = B_2 v_2 = B_3 v_3 = B_4 v_4 = 0$.

Hence, the solution of the completely inhomogeneous problem can be written as:

$$u(x) = \int_0^1 G(x,\xi) f(\xi) d\xi + \frac{\alpha_2}{B_2 v_1} v_1(x) + \frac{\alpha_1}{B_1 v_2} v_2(x) + \frac{\alpha_4}{B_4 v_3} v_3(x) + \frac{\alpha_3}{B_3 v_4} v_4(x)\,. \qquad (2.55)$$

In the following, we give the solutions of completely non-homogenous problems corresponding to the set of boundary conditions constructed before:

a) Hinged Both Ends

$$\frac{d^4 u}{dx^4} = f(x), \qquad 0 \le x \le 1\,,$$

$$\begin{aligned} u(0) &= \alpha_1, u''(0) = \beta_1, \\ u(1) &= \alpha_2, u''(1) = \beta_2. \end{aligned}$$

The complete solution is:

$$\begin{aligned} u(x) &= v(x) + w(x) \\ &= \int_0^x \left[\frac{1}{3}\xi(1-\xi)(1-x) + \frac{1}{6}\xi(x-1)^3 \right] f(\xi) d\xi \\ &\quad + \int_x^1 \left[\frac{1}{6}\xi(1-\xi)x + \frac{1}{6}(\xi-1)x^3 \right] f(\xi) d\xi + \alpha_1 + \left(\alpha_2 - \alpha_1 - \frac{1}{3}\beta_1 - \frac{1}{6}\beta_2 \right) x \\ &\quad + \frac{1}{2}\beta_1 x^2 + \frac{1}{6}(\beta_2 - \beta_1) x^3. \end{aligned} \qquad (2.56)$$

b) Clamped Both Ends

$$\frac{d^4u}{dx^4} = f(x), \qquad 0 \le x \le 1,$$

$$u(0) = \alpha_1, u'(0) = \beta_1,$$
$$u(1) = \alpha_2, u'(1) = \beta_2.$$

The complete solution is:

$$\begin{aligned} u(x) &= v(x) + w(x) \\ &= \int_0^x \left[\frac{1}{2}\xi^2(1-\xi)(1-x)^2 + \frac{1}{6}\xi^2(2\xi-3)(x-1)^3\right] f(\xi)d\xi \\ &\quad + \int_x^1 \left[\frac{1}{2}\xi(\xi^2-2\xi+1)x^2 + \frac{1}{6}(3\xi^2-2\xi^3-1)x^3\right] f(\xi)d\xi \\ &\quad + \alpha_1 + \beta_1 x + \left(-\beta_2 - 2\beta_1 - 3\alpha_1 + 3\alpha_2\right)x^2 + \left(2\alpha_1 - 2\alpha_2 + \beta_1 + \beta_2\right)x^3. \end{aligned} \tag{2.57}$$

c) One End Clamped-One Free

$$\frac{d^4u}{dx^4} = f(x), \qquad 0 \le x \le 1,$$

$$u(0) = \alpha_1, u'(0) = \beta_1,$$
$$u''(1) = \alpha_2, u'''(1) = \beta_2.$$

The complete solution is:

$$\begin{aligned} u(x) &= v(x) + w(x) \\ &= \int_0^x \left[\frac{1}{6}\xi^2(3-\xi) - \frac{1}{2}\xi^2(1-x)\right] f(\xi)d\xi \\ &\quad + \int_x^1 \left[\frac{1}{2}\xi x^2 - \frac{1}{6}x^3\right] f(\xi)d\xi + \alpha_1 + \beta_1 x + \frac{1}{2}\left(\alpha_2 - \beta_1\right)x^2 + \frac{1}{6}\beta_2 x^3. \end{aligned} \tag{2.58}$$

CHAPTER 3

NON-HOMOGENOUS EULER-BERNOULLI BEAM EQUATION

We have seen that the governing equation of the Euler-Bernoulli beam can be written as the following form:

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2u}{dx^2}\right) = f(x), \tag{3.1}$$

where E is the Young's modulus, I is the moment of inertia and $f(x)$ is the external load (force).

In practical situations, the beam may have the variable elastic properties due to inhomogeneities or mixture of materials. In such a case EI will be a function of x, so that it can be written as the following form:

$$EI = q(x).$$

The Equation (3.1) then becomes as following:

$$\frac{d^2}{dx^2}\left(q(x)\frac{d^2u}{dx^2}\right) = f(x). \tag{3.2}$$

3.1 Perturbation Problem

We assume that the variation in the elastic properties is very small so that we can write $q(x)$ as the following form:

$$q(x) = q_0 + \varepsilon q_1(x) + \ldots \quad . \tag{3.3}$$

So, that Equation (3.2) becomes:

$$\frac{d^2}{dx^2}\left(\left(q_0 + \varepsilon q_1(x) + \ldots\right)\frac{d^2u}{dx^2}\right) = f(x), \tag{3.4}$$

or,

$$q_0\frac{d^4u}{dx^4} + \varepsilon\frac{d^2}{dx^2}\left(q_1(x)\frac{d^2u}{dx^2} + \ldots\right) = f(x), \tag{3.5}$$

or,

$$q_0\frac{d^4u}{dx^4} + \varepsilon\frac{d}{dx}\left(q_1'(x)\frac{d^2u}{dx^2} + q_1(x)\frac{d^3u}{dx^3} + \ldots\right) = f(x), \tag{3.6}$$

or,

$$q_0\frac{d^4u}{dx^4} + \varepsilon\frac{d}{dx}\left(q_1'(x)\frac{d^2u}{dx^2} + q_1(x)\frac{d^3u}{dx^3}\right) + \ldots = f(x), \tag{3.7}$$

or,

$$q_0\frac{d^4u}{dx^4} + \varepsilon\left[\frac{d}{dx}\left(q_1'(x)\frac{d^2u}{dx^2} + \right) + \frac{d}{dx}\left(q_1(x)\frac{d^3u}{dx^3}\right)\right] + \ldots = f(x), \tag{3.8}$$

or,

$$q_0 \frac{d^4u}{dx^4} + \varepsilon\left[q_1''(x)\frac{d^2u}{dx^2} + q_1'(x)\frac{d^3u}{dx^3} + q_1'(x)\frac{d^3u}{dx^3} + q_1(x)\frac{d^4u}{dx^4} \right] + \ldots = f(x)\,, \qquad (3.9)$$

or by simply,

$$q_0 \frac{d^4u}{dx^4} + \varepsilon\left[q_1''(x)\frac{d^2u}{dx^2} + 2q_1'(x)\frac{d^3u}{dx^3} + q_1(x)\frac{d^4u}{dx^4} \right] + \ldots = f(x)\,. \qquad (3.10)$$

Now, let us assume that the function $u(x)$ can be written as:

$$u(x) = u_0(x) + \varepsilon u_1(x) + \ldots \quad . \qquad (3.11)$$

By substituting Eq. (3.11) in Eq. (3.10) we have the following form:

$$q_0 \frac{d^4u_0}{dx^4} + \varepsilon\left[q_0 \frac{d^4u_1}{dx^4} + q_1(x)\frac{d^4u_0}{dx^4} + 2q_1'(x)\frac{d^3u_0}{dx^3} + q_1''(x)\frac{d^2u_0}{dx^2} \right] + \ldots = f(x)\,. \qquad (3.12)$$

Now, we compare the coefficients of like powers of ε, we have:

For the coefficient $\varepsilon^{(0)}$, we obtain the following:

$$q_0 \frac{d^4u}{dx^4} = f(x)\,, \qquad (3.13)$$

or,

$$\frac{d^4u_0}{dx^4} = F_0(x) \qquad \text{(unperturbed case)}, \qquad (3.14)$$

where $F_0(x) = \dfrac{f(x)}{q_0}$, provided that $q_0 \neq 0$.

For the coefficient $\varepsilon^{(1)}$, we obtain the following:

$$q_0(x)\frac{d^4u_1}{dx^4}+q_1(x)\frac{d^4u_0}{dx^4}+2q_1'(x)\frac{d^3u_0}{dx^3}+q_1''(x)\frac{d^2u_0}{dx^2}=0\,, \qquad (3.15)$$

or,

$$\frac{d^4u_1}{dx^4}=\frac{-1}{q_0}\left(q_1(x)\frac{d^4u_0}{dx^4}+2q_1'(x)\frac{d^3u_0}{dx^3}+q_1''(x)\frac{d^2u_0}{dx^2}\right) \qquad \text{(perturbed case).} \qquad (3.16)$$

The solution of Eq. (3.14), under the prescribed boundary conditions in the three cases that constructed before is:

$$u_0(x)=\int_0^1 G(x,\xi)F_0(\xi)d\xi\,. \qquad (3.17)$$

Also, the solution of Eq. (3.16) can be written as:

$$u_1(x)=\int_0^1 G(x,\xi)F_1(\xi)d\xi\,, \qquad (3.18)$$

where $F_1(x)=\frac{-1}{q_0}\left(q_1(x)\frac{d^4u_0}{dx^4}+2q_1'(x)\frac{d^3u_0}{dx^3}+q_1''(x)\frac{d^2u_0}{dx^2}\right)$ and $G(x,\xi)$ is Green's function for each case that constructed before.

Thus, $u(x)=u_0(x)+\varepsilon u_1(x)+\ldots$ is found correct solution to the first power of ε.

In order to compute the solution of the perturbed problem, we need to evaluate the integral in Equation (3.18). We shall do this for two cases of interest. The first case is a beam of concrete and the second case is steel beam. The Young's modulus and the moment of inertia for these are given below:

Table 3.1: ***Young's modules and moment of inertia of concrete and steel beams***

Materials	Young's Modulus E	Moment of Inertia I	$q_0 = EI$
Concrete	$17*10^9\ Nm^{-2}$	$\frac{1}{12}bh^3 = 0.25\,m^4$	$\frac{17*10^9*0.25}{12}\,Nm^2$
Steel	$210*10^9\ Nm^{-2}$	$\frac{1}{12}bh^3 = 0.25\,m^4$	$\frac{210*10^9*0.25}{12}\,Nm^2$

Here, we have $I = \frac{1}{12}bh^3$, where $b = 0.25\,m$ is the width of the beam and $h = 1\,m$ is the height of the beam.

In the following, we present the solutions of perturbed problem in cases of concrete and steel beams respectively and we note that in each case, we take $q_1(x) = 10^{10}x$ as a linear variation and so we have $q_1'(x) = 10^{10}$ and $q_1''(x) = 0$, so we have:

a. Hinged Both Ends

i. Concrete Beam

We consider a completely non-homogenous problem for a concrete beam with hinged boundary conditions.

In this case, as we derived Green's function $G(x,\xi)$. The solution of unperturbed problem is:

$$u(x) = u_0(x) + \varepsilon u_1(x) + ...$$

$$= \int_0^1 G(x,\xi)F_0(\xi)d\xi + \varepsilon\left[\int_0^1 G(x,\xi)F_1(\xi)d\xi\right] + ..., \qquad (3.19)$$

where

$$u_0(x) = \int_0^1 G(x,\xi)F_0(\xi)d\xi\ , \qquad (3.20)$$

$$u_1(x)=\int_0^1 G(x,\xi)F_1(\xi)d\xi\ , \qquad (3.21)$$

$$F_0(\xi)=\xi^2\ , \qquad (3.22)$$

$$F_1(\xi)=\frac{-1}{q_0}\left(q_1(\xi)\frac{d^4u_0}{dx^4}+2q_1'(\xi)\frac{d^3u_0}{dx^3}+q_1''(\xi)\frac{d^2u_0}{dx^2}\right) \qquad (3.23)$$

and $$G(x,\xi)=\begin{cases}\frac{1}{6}\xi(1-\xi)x+\frac{1}{6}(\xi-1)x^3, & 0\le x<\xi\le 1,\\ \frac{1}{3}\xi(1-\xi)(1-x)+\frac{1}{6}\xi(x-1)^3, & 0\le \xi<x\le 1.\end{cases} \qquad (3.24)$$

The following graph is the solutions of $u(x)$ in different cases of ε:

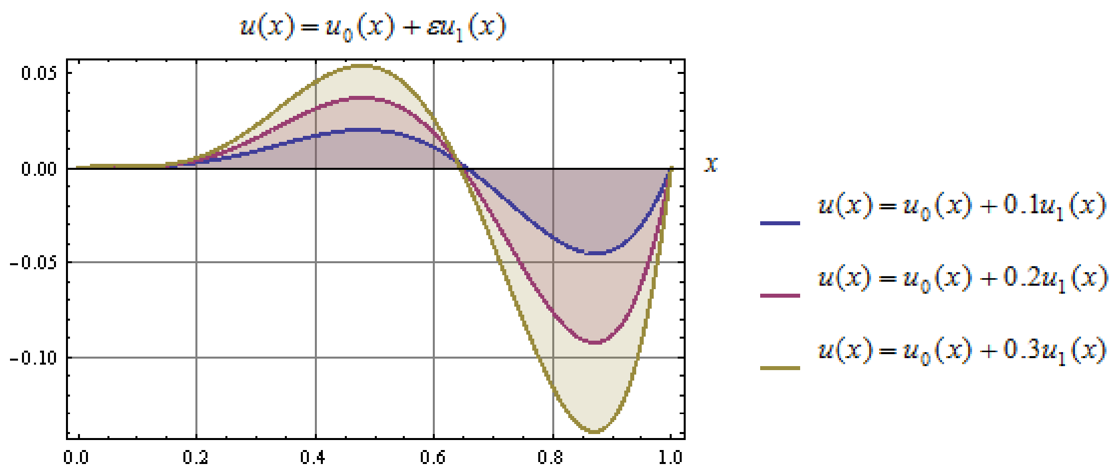

Figure 3.1: ***Solutions of concrete beam with hinged boundary conditions***

ii. Steel Beam

We consider a completely non-homogenous problem for steel beam with hinged boundary conditions.

In this case, as we derived Green's function $G(x,\xi)$. The solution of unperturbed problem is:

$$u(x)=u_0(x)+\varepsilon u_1(x)+...$$

$$= \int_0^1 G(x,\xi)F_0(\xi)d\xi + \varepsilon\left[\int_0^1 G(x,\xi)F_1(\xi)d\xi\right] + \ldots, \qquad (3.25)$$

where $u_0(x)$, $u_1(x)$, $F_0(\xi)$, $F_1(\xi)$ and $G(x,\xi)$ are given in equations (3.20), (3.21), (3.22), (3.23) and (3.24 or 2.14) respectively.

The following graph is the solutions of $u(x)$ in different cases of ε:

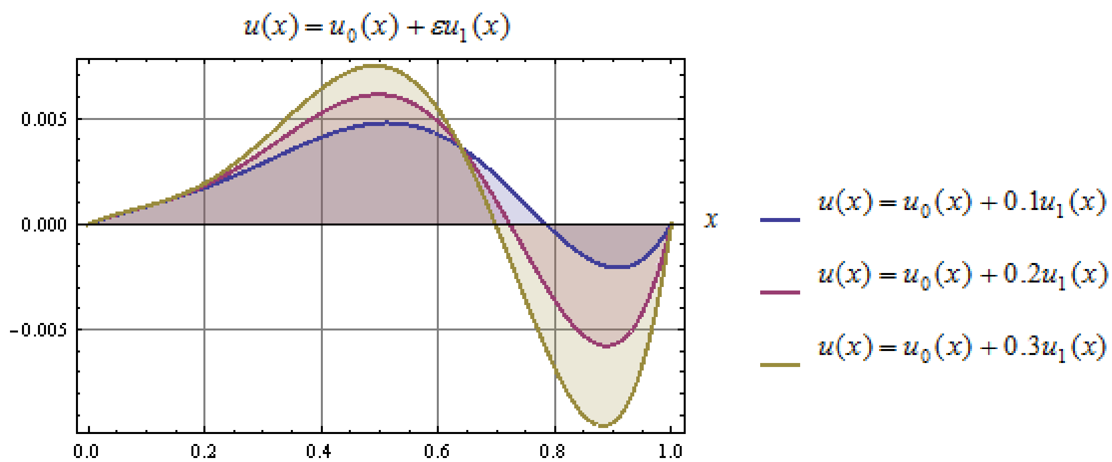

Figure 3.2: ***Solutions of steel beam with hinged boundary conditions***

b. Clamped Both Ends

i. Concrete Beam

We consider a completely non-homogenous problem for a concrete beam with clamped boundary conditions.

In this case, as we derived Green's function $G(x,\xi)$. The solution of unperturbed problem is:

$$u(x) = u_0(x) + \varepsilon u_1(x) + \ldots$$

$$= \int_0^1 G(x,\xi)F_0(\xi)d\xi + \varepsilon\left[\int_0^1 G(x,\xi)F_1(\xi)d\xi\right] + \ldots, \qquad (3.26)$$

where $u_0(x)$, $u_1(x)$, $F_0(\xi)$, $F_1(\xi)$ and $G(x,\xi)$ are given in equations (3.20), (3.21), (3.22), (3.23) and (2.28) respectively.

The following graph is the solutions of $u(x)$ in different cases of ε:

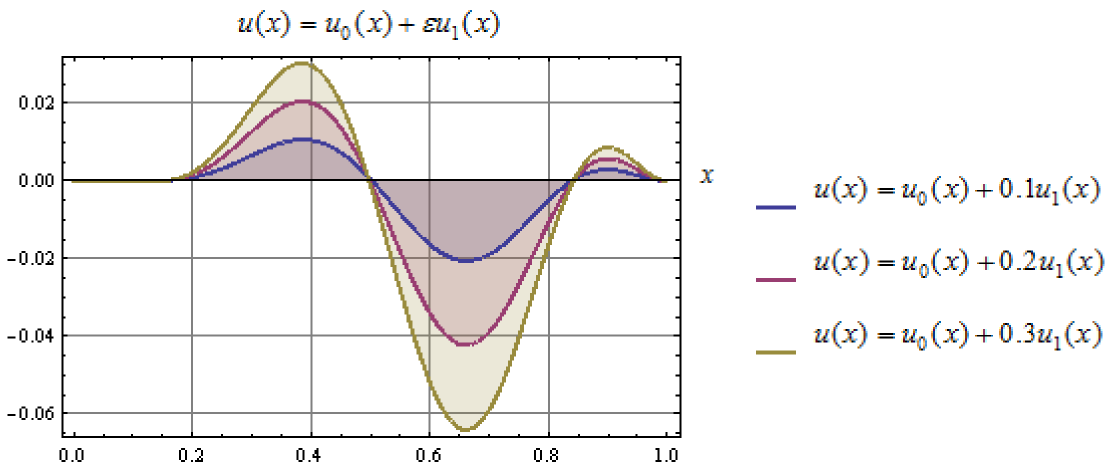

Figure 3.3: ***Solutions of concrete beam with clamped boundary conditions***

ii. Steel Beam

We consider a completely non-homogenous problem for steel beam with clamped boundary conditions.

In this case, as we derived Green's function $G(x,\xi)$. The solution of unperturbed problem is:

$$u(x) = u_0(x) + \varepsilon u_1(x) + \ldots$$

$$= \int_0^1 G(x,\xi)F_0(\xi)d\xi + \varepsilon\left[\int_0^1 G(x,\xi)F_1(\xi)d\xi\right] + \ldots, \qquad (3.27)$$

where $u_0(x)$, $u_1(x)$, $F_0(\xi)$, $F_1(\xi)$ and $G(x,\xi)$ are given in equations (3.20), (3.21), (3.22), (3.23) and (2.28) respectively.

The following graph is the solutions of $u(x)$ in different cases of ε:

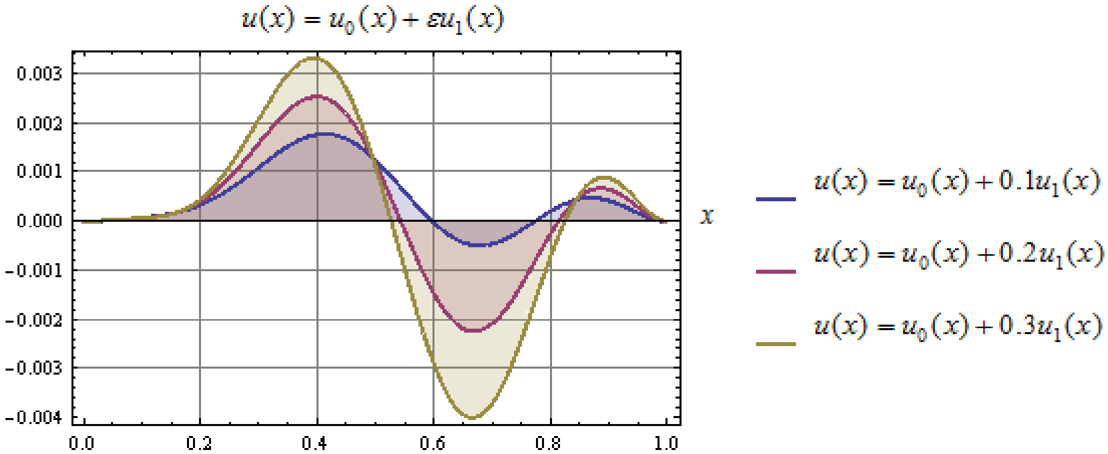

Figure 3.4: ***Solutions of steel beam with clamped boundary conditions***

c. Clamped at One End-One Free

i. Concrete Beam

We consider a completely non-homogenous problem for a concrete beam with clamped at one end-one free boundary conditions.

In this case, as we derived Green's function $G(x,\xi)$. The solution of unperturbed problem is:

$$u(x) = u_0(x) + \varepsilon u_1(x) + \ldots$$

$$= \int_0^1 G(x,\xi)F_0(\xi)d\xi + \varepsilon\left[\int_0^1 G(x,\xi)F_1(\xi)d\xi\right] + \ldots, \qquad (3.23)$$

where $u_0(x)$, $u_1(x)$, $F_0(\xi)$, $F_1(\xi)$ and $G(x,\xi)$ are given in equations (3.20), (3.21), (3.22), (3.23) and (2.39) respectively.

The following graph is the solutions of $u(x)$ in different cases of ε:

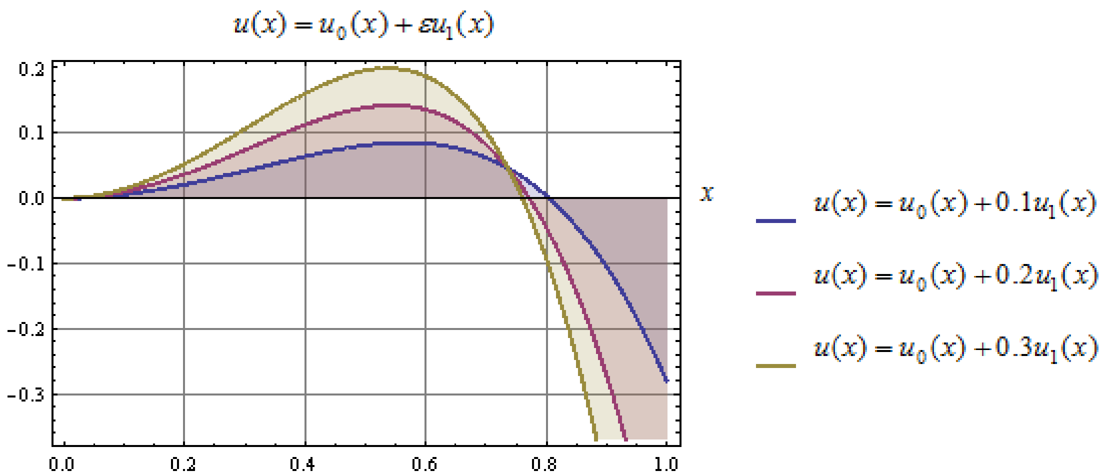

Figure 3.5: ***Solutions of concrete beam with clamped at one end-one free***

ii. Steel Beam

We consider a completely non-homogenous problem for steel beam with clamped at one end-one free boundary conditions.

In this case, as we derived Green's function $G(x,\xi)$. The solution of unperturbed problem is:

$$u(x)=u_0(x)+\varepsilon u_1(x)+...$$

$$=\int_0^1 G(x,\xi)F_0(\xi)d\xi+\varepsilon\left[\int_0^1 G(x,\xi)F_1(\xi)d\xi\right]+...,\qquad (3.24)$$

where $u_0(x)$, $u_1(x)$, $F_0(\xi)$, $F_1(\xi)$ and $G(x,\xi)$ are given in equations (3.20), (3.21), (3.22), (3.23) and (2.39) respectively.

The following graph is the solutions of $u(x)$ in different cases of ε:

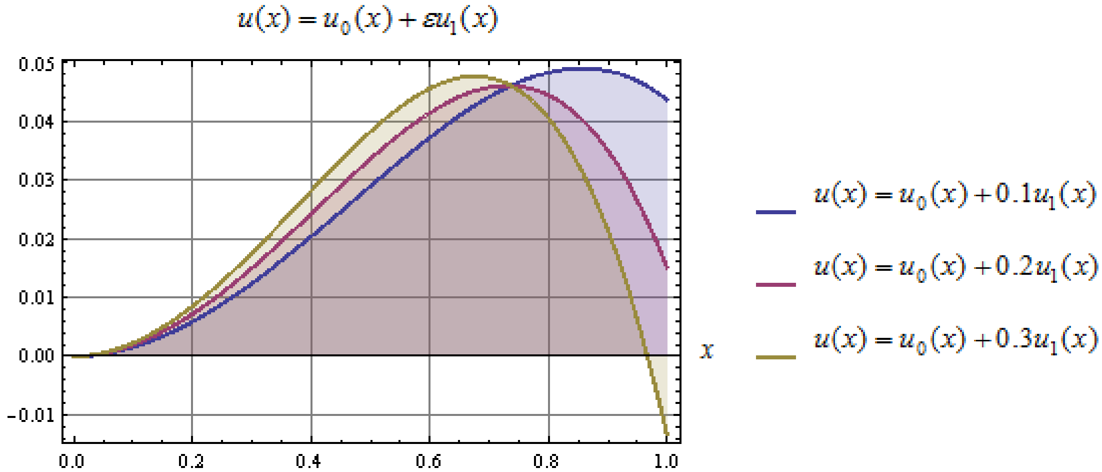

Figure 3.6: ***Solutions of steel beam with clamped at one end-one free***

CHAPTER 4

DYNAMIC MODEL FOR NON-HOMOGENOUS EULER-BERNOULLI BEAM EQUATION

In this chapter, we consider the transverse vibrations of an Euler-Bernoulli beam with non-homogenous elastic properties. We use the perturbation theory to study this problem. We have seen in Chapter 1, that the fourth-order differential linear expression $\frac{d^4}{dx^4}$ defines a self-adjoint operator corresponding to the boundary conditions considered in earlier chapters. As a consequence, we have the eigenvalue of the problem $\frac{d^4u}{dx^4} = \lambda u$ are real and the eigenfunctions corresponding to the distinct eigenvalues are orthogonal.

4.1 Eigenvalue Perturbation Problem

Suppose that $L(\varepsilon)$ is a self-adjoint operator defined in a real *Hilbert Space* H. We assume that $L(\varepsilon)$ depends continuously on ε, i.e.

$$\lim_{\Delta\varepsilon \to 0} \left\| L(\varepsilon + \Delta\varepsilon) - L(\varepsilon) \right\| = 0. \tag{4.1}$$

We would like to relate the spectrum of the perturbed operator $L(\varepsilon)$ to the presumable known spectrum of the base operator $L = L(0)$.

Theorem 4.1

Let $\lambda_n \neq 0$ *be a simple eigenvalue of the operator* L *with normalized eigenfunctions* e_n. *Then in some neighborhood of* $\varepsilon \to 0$, *there exists an eigenpair* $(\lambda(\varepsilon), e(\varepsilon))$ *of the perturbed eigenvalue problem* $L(\varepsilon)u(x,\varepsilon) = \lambda(\varepsilon)u(x,\varepsilon)$ *with the following properties:*

$$\lim_{\varepsilon \to 0} \lambda(\varepsilon) = \lambda_n \text{ and } \lim_{\varepsilon \to 0} e(\varepsilon) = e_n .$$

Before we go to the fourth-order problem, we give the following illustrative example.

Consider the boundary value problem:

$$-u''(x,\varepsilon) + (1+\varepsilon x^2)u(x,\varepsilon) = \mu u(x,\varepsilon), \qquad 0 < x < 1, \qquad (4.2)$$

with boundary condition:

$$u(0,\varepsilon) = 0, u(1,\varepsilon) = 0 . \qquad (4.2a)$$

When $\varepsilon = 0$, we have:

$$-u''(x,0) = (\mu - 1)u(x,0), \qquad 0 < x < 1, \qquad (4.3)$$

with boundary conditions:

$$u(0,0) = 0, u(1,0) = 0 . \qquad (4.3a)$$

The eigenvalues and the normalized eigenfunctions of problem (4.3) equipped with boundary conditions given in (4.3a) are given below respectively:

$$\mu_n = n^2\pi^2 + 1, \quad n = 1,2,\ldots , \qquad (4.4)$$

$$e_n(x)=\sqrt{2}\sin(n\pi x),\ \ n=1,2,... \quad . \qquad (4.5)$$

From Theorem 4.1, there exists an eigenpair $(\lambda(\varepsilon),u(\varepsilon))$ with property $\lambda(\varepsilon)\to\lambda_n, u(x)\to e_n(x)$ as $\varepsilon\to 0$, so that there is a branch $\left(\mu(\varepsilon),u(x,\varepsilon)\right)\to\left(\mu_n,e_n(x)\right)$ as $\varepsilon\to 0$.

As the operator in Equation (4.2) depends continuously on ε, we can differentiate (4.2) with respect ε to obtain the following:

$$-u_\varepsilon''(x,\varepsilon)+(1+\varepsilon x^2)u_\varepsilon(x,\varepsilon)-\mu u_\varepsilon(x,\varepsilon)=\mu_\varepsilon u(x,\varepsilon)-x^2u(x,\varepsilon)\ , \qquad (4.6)$$

$$u_\varepsilon(0,\varepsilon)=0, u_\varepsilon(0,\varepsilon)=0. \qquad (4.6a)$$

For $\varepsilon=0$, we get an inhomogeneous equation for $u_\varepsilon(x,\varepsilon)$ as follows:

$$-u_\varepsilon''(x,0)+u_\varepsilon(x,0)-\mu_n u_\varepsilon(x,0)=\mu_\varepsilon(0)e_n(x)-x^2e_n(x). \qquad (4.7)$$

Equation (4.7) cannot be solved for $u_\varepsilon(x,0)$ unless $\mu_\varepsilon(0)$ is known and it must satisfy the following solvability condition:

$$\langle\mu_\varepsilon(0)e_n(x)-x^2e_n(x),e_n(x)\rangle=0. \qquad (4.8)$$

Equation (4.8) leads us to have:

$$\mu_\varepsilon(0)=\int_0^1 x^2e_n{}^2(x)dx, \qquad (4.9)$$

and so we have:

$$\mu(\varepsilon) = \mu_n + \varepsilon\int_0^1 x^2 e_n{}^2(x)dx + ... \quad . \qquad (4.10)$$

Furthermore, there is a unique solution $w(x)$, which is orthogonal to the normalized eigenfunctions $e_n(x)$ given by:

$$w(x) = -\sum_{m\neq n} \frac{\langle x^2 e_n(x), e_n(x)\rangle}{(m^2 - n^2)\pi^2} e_m(x), \qquad (4.11)$$

where $e_n(x) = \sqrt{2}\sin(m\pi x)$.

Thus, we have the normalized eigenfunctions take the following form given by (Ivar Stackgold[14]):

$$u_n(x,\varepsilon) = e_n(x) + \varepsilon w(x), \qquad (4.12)$$

where the normalization $\langle u_n, e_n\rangle = 1$ is used.

4.2 Transverse Vibration of Euler-Bernoulli Beam Equation as an Eigenvalue Problem

Consider the following problem:

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 u(x,t)}{\partial x^2}\right) = \rho\frac{\partial^2 u(x,t)}{\partial t^2}, \qquad 0 < x < 1,\ \ t > 0 \quad ,\ \rho > 0. \qquad (4.13)$$

When $EI = q_0$ (constant), then we have:

$$\frac{\partial^4 u(x,t)}{\partial x^4} = \frac{\rho}{q_0}\frac{\partial^2 u(x,t)}{\partial t^2} \qquad \text{(provided that } q_0 \neq 0\text{)} . \qquad (4.14)$$

We assume that the vibrations are time harmonic, i.e.

$$u(x,t) = u(x)e^{i\omega t}, \qquad (4.15)$$

where ω is the angular frequency.

By substituting Eq. (4.15) in Eq. (4.14), we get:

$$\frac{d^4u(x)}{dx^4} = -\frac{\rho\omega^2}{q_0}u(x), \text{ or}$$

$$\frac{d^4u(x)}{dx^4} + \lambda u(x) = 0, \quad 0 < x < 1, \qquad (4.16)$$

where $\lambda = \dfrac{\rho\omega^2}{q_0}$.

Thus, we see that the transverse vibration of an elastic Euler-Bernoulli beam can be cast into an equivalent eigenvalue problem of the type given in Eq. (4.16).

In the following examples, we deal with the eigenvalue problem (4.16) in the three cases of hinged, clamped and clamped-free boundary conditions.

Example 4.1 (Beam with Both Ends Hinged)

In this case, we have the following eigenvalue problem:

$$\begin{cases} \dfrac{d^4u}{dx^4} + \lambda u = 0, \quad 0 < x < 1, \\ \text{with hinged boundary conditions}, \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \qquad (4.17)$$

We can easily see that $\lambda = 0$ is not an eigenvalue because we get a trivial solution i.e. $u \equiv 0$. Also, as the eigenvalues are real and positive we now assume that $\lambda = \alpha^4$, for some $\alpha \in R$, so that the characteristic equation is given as:

$$m^4 + \alpha^4 = 0, \qquad (4.18)$$

so the roots of Eq. (4.18) are given as following:

$$\begin{aligned} m_1 &= \alpha(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}), m_2 = \alpha(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}), \\ m_3 &= \alpha(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}), m_4 = \alpha(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}). \end{aligned} \qquad (4.19)$$

Thus, the general solution of the problem (4.17) is given by:

$$u(x) = e^{\frac{\alpha}{\sqrt{2}}x}[c_1 \cos(\frac{\alpha}{\sqrt{2}}x) + c_2 \sin(\frac{\alpha}{\sqrt{2}}x)] + e^{-\frac{\alpha}{\sqrt{2}}x}[c_3 \cos(\frac{\alpha}{\sqrt{2}}x) + c_4 \sin(\frac{\alpha}{\sqrt{2}}x)]. \qquad (4.20)$$

Now, using the hinged boundary conditions, we end up with the following values $\lambda_n = \alpha_n{}^4 = (\sqrt{2}n\pi)^4 = 4n^4\pi^4, n = 1,2,3,...$ as they are the eigenvalues of the supply-supported beam (hinged beam with both ends).

Hence, the corresponding normalized eigenfunctions are given by:

$$u_n(x) = \sqrt{2}\sin(n\pi x), \qquad n = 1,2,3,... \ . \qquad (4.21)$$

Example 4.2 (Beam with Both Ends Clamped)

In this case, we have the following eigenvalue problem:

$$\begin{cases} \dfrac{d^4u}{dx^4} + \lambda u = 0, \quad 0 < x < 1, \\ \textit{with clamped boundary conditions}, \\ u(0) = 0, u'(0) = 0, u(1) = 0, u'(1) = 0. \end{cases} \tag{4.22}$$

Also, we can easily see that $\lambda = 0$ is not an eigenvalue because we get a trivial solution i.e. $u \equiv 0$. Furthermore, as the eigenvalues are real and positive we now assume that $\lambda = \beta^4$, for some $\beta \in R$, so that the characteristic equation is given as:

$$m^4 + \beta^4 = 0, \tag{4.23}$$

so the roots of Eq. (4.23) are given as following:

$$\begin{aligned} m_1 &= \beta(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}), m_2 = \beta(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}), \\ m_3 &= \beta(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}), m_4 = \beta(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}). \end{aligned} \tag{4.24}$$

The general solution of the problem (4.22) is given by:

$$u(x) = e^{\frac{\beta}{\sqrt{2}}x}[c_1\cos(\frac{\beta}{\sqrt{2}}x) + c_2\sin(\frac{\beta}{\sqrt{2}}x)] + e^{-\frac{\beta}{\sqrt{2}}x}[c_3\cos(\frac{\beta}{\sqrt{2}}x) + c_4\sin(\frac{\beta}{\sqrt{2}}x)]. \tag{4.25}$$

Now, using the clamped boundary conditions, we get the smallest eigenvalue is given as $\lambda = \beta^4 = (4.7300407)^4 = 500.56655$.

Example 4.3 (Beam with Ends Clamped-Free)

In this case, we have the following eigenvalue problem:

$$\begin{cases} \dfrac{d^4u}{dx^4} + \lambda u = 0, \quad 0 < x < 1, \\ \text{with clamped} - \text{free boundary conditions}, \\ u(0) = 0, u'(0) = 0, u''(1) = 0, u'''(1) = 0. \end{cases} \tag{4.26}$$

Again also, we can easily see that $\lambda = 0$ is not an eigenvalue because we get a trivial solution i.e. $u \equiv 0$. In addition, as the eigenvalues are real and positive we now assume that $\lambda = \gamma^4$, for some $\gamma \in R$, so that the characteristic equation is given as:

$$m^4 + \gamma^4 = 0, \tag{4.27}$$

so the roots of Eq. (4.27) are given as following:

$$\begin{aligned} m_1 &= \gamma(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}), m_2 = \gamma(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}), \\ m_3 &= \gamma(\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}), m_4 = \gamma(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}). \end{aligned} \tag{4.28}$$

The general solution of the problem (4.26) is given by:

$$u(x) = e^{\frac{\gamma}{\sqrt{2}}x}[c_1 \cos(\frac{\gamma}{\sqrt{2}}x) + c_2 \sin(\frac{\gamma}{\sqrt{2}}x)] + e^{-\frac{\gamma}{\sqrt{2}}x}[c_3 \cos(\frac{\gamma}{\sqrt{2}}x) + c_4 \sin(\frac{\gamma}{\sqrt{2}}x)]. \tag{4.29}$$

Using the clamped-free boundary conditions, we end up with the smallest eigenvalue $\lambda = \gamma^4 = (1.8751)^4 = 12.3623$.

4.3 Vibration of Non-Homogenous Euler-Bernoulli Beam Equation

In this Section, we consider the free vibration of non-homogeneous Euler-Bernoulli Beam. As before, we assume that the variation of elastic properties is small, so that we can write the following:

$$EI(x) = q(x) = q_0 + \varepsilon q_1(x) + \dots, \qquad (4.30)$$

where q_0 is a non-zero constant and $q_1(x)$ is an infinite differentiable function of x.

The equation of vibrating beam can be written as:

$$\frac{d^2}{dx^2}\left[\left(q_0 + \varepsilon q_1(x)\right)\frac{d^2u(x,\varepsilon)}{dx^2} + \dots\right] = \lambda(\varepsilon)u(x,\varepsilon), \qquad 0 < x < 1, \qquad (4.31)$$

with appropriate boundary conditions.

Here, we assume that the beam is hinged at both ends so, we have:

$$u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \qquad (4.32)$$

We notice that for $\varepsilon = 0$, the Equation (4.31) reduces to the one studied in Section 4.2, having the eigenvalues $\lambda_n = 4n^4\pi^4, n = 1,2,3,...$ with corresponding normalized eigenfunctions $e_n(x) = \sqrt{2}\sin(n\pi x), n = 1,2,3,\dots$.

As before, the Equation (4.31) can be written as follows:

For all $0 < x < 1$, we have:

$$u^{(4)}(x,\varepsilon) + \varepsilon\left[\frac{q_1(x)}{q_0}u^{(4)}(x,\varepsilon) + 2\frac{q_1'(x)}{q_0}u'''(x,\varepsilon) + \frac{q_1''(x)}{q_0}u''(x,\varepsilon)\right] + \dots = \lambda(\varepsilon)u(x,\varepsilon), \qquad (4.33)$$

with hinged boundary conditions given as:

$$u(0,\varepsilon)=0, u''(0,\varepsilon)=0, u(1,\varepsilon)=0, u''(1,\varepsilon)=0\,. \qquad \text{(4.33i)}$$

Now, we have the eigenpair of the perturbed problem (4.33).

We assume that we can differentiate Equation (4.33) with respect to ε, so we get:

$$\begin{aligned}
&u_\varepsilon^{(4)}(x,\varepsilon)+\Big[\frac{q_1(x)}{q_0}u^{(4)}(x,\varepsilon)+2\frac{q_1'(x)}{q_0}u'''(x,\varepsilon)+\frac{q_1''(x)}{q_0}u''(x,\varepsilon)\Big]\\
&+\varepsilon\Big[\frac{q_1(x)}{q_0}u_\varepsilon^{(4)}(x,\varepsilon)+2\frac{q_1'(x)}{q_0}u_\varepsilon'''(x,\varepsilon)+\frac{q_1''(x)}{q_0}u_\varepsilon''(x,\varepsilon)\Big]\\
&=\lambda_\varepsilon(\varepsilon)u(x,\varepsilon)+\lambda(\varepsilon)u_\varepsilon(x,\varepsilon).
\end{aligned} \qquad \text{(4.34)}$$

Now, the eigenvalues $\lambda(\varepsilon)$ and the normalized eigenfunctions $u(x,\varepsilon)$ for $\varepsilon\to 0$ become $\lambda(0)=\lambda_n=4n^4\pi^4$ and $u(x,0)=u_n(x)=\sqrt{2}\sin(n\pi x)$ respectively, where $n=1,2,3,\ldots$. Thus, Equation (4.34) can be written as following:

$$\begin{aligned}
u_\varepsilon^{(4)}(x,0)+\lambda_n u_\varepsilon(x,0) = 2\lambda_\varepsilon(0)\sin(n\pi x)-\Big[&4\sqrt{2}n^4\pi^4\,\frac{q_1(x)}{q_0}\sin(n\pi x)\\
&-8n^3\pi^3\,\frac{q_1'(x)}{q_0}\cos(n\pi x)-2\sqrt{2}n^2\pi^2\,\frac{q_1''(x)}{q_0}\sin(n\pi x)\Big],
\end{aligned} \qquad \text{(4.35)}$$

with the boundary conditions given as below:

$$u_\varepsilon(0,0)=0, u_\varepsilon''(0,0)=0, u_\varepsilon(1,0)=0, u_\varepsilon''(1,0)=0\,. \qquad \text{(4.35i)}$$

As $\lambda_\varepsilon(0)$ is unknown, we cannot solve problem (4.35) directly. As the corresponding homogenous problem has non-trivial solution $e_n(x)$, the Equation (4.35) with the boundary conditions given by (4.35i) has solution if the following solvability condition is satisfied:

$$\begin{aligned}&\langle\sqrt{2}\lambda_{\varepsilon}(0)\sin(n\pi x)-\Big[4\sqrt{2}n^4\pi^4\frac{q_1(x)}{q_0}\sin(n\pi x)-8n^3\pi^3\frac{q_1'(x)}{q_0}\cos(n\pi x)\\&-2\sqrt{2}n^2\pi^2\frac{q_1''(x)}{q_0}\sin(n\pi x)\Big],\sqrt{2}\sin(n\pi x)\rangle=0.\end{aligned}\tag{4.36}$$

Equation (4.36) leads us to have:

$$\begin{aligned}\int_0^1 2\lambda_{\varepsilon}(0)\sin^2(n\pi x)dx-\int_0^1\Big[4\sqrt{2}n^4\pi^4\frac{q_1(x)}{q_0}\sin(n\pi x)-8n^3\pi^3\frac{q_1'(x)}{q_0}\cos(n\pi x)\\-2\sqrt{2}n^2\pi^2\frac{q_1''(x)}{q_0}\sin(n\pi x)\Big]\sqrt{2}\sin(n\pi x)dx=0,\end{aligned}\tag{4.37}$$

which gives the value of $\lambda_{\varepsilon}(0)$ as in the following:

$$\begin{aligned}\lambda_{\varepsilon}(0)=\int_0^1\Big[8n^4\pi^4\frac{q_1(x)}{q_0}\sin^2(n\pi x)-8\sqrt{2}n^3\pi^3\frac{q_1'(x)}{q_0}\sin(2n\pi x)\\-4n^2\pi^2\frac{q_1''(x)}{q_0}\sin^2(n\pi x)\Big]dx.\end{aligned}\tag{4.38}$$

Therefore, the general form of the eigenvalues for vibrating non-homogenous beam with hinged boundary conditions is given by:

$$\lambda(\varepsilon)=\lambda_n+\varepsilon\lambda_{\varepsilon}(0)+\dots\ ,$$

so, we have:

$$\begin{aligned}\lambda(\varepsilon)=4n^4\pi^4+\varepsilon\int_0^1\Big[8n^4\pi^4\frac{q_1(x)}{q_0}\sin^2(n\pi x)-8\sqrt{2}n^3\pi^3\frac{q_1'(x)}{q_0}\sin(2n\pi x)\\-4n^2\pi^2\frac{q_1''(x)}{q_0}\sin^2(n\pi x)\Big]dx+\dots\ .\end{aligned}\tag{4.39}$$

The smallest eigenvalue when $n=1$ is:

$$\lambda(\varepsilon)=4\pi^4+\varepsilon\int_0^1\Big[8\pi^4\frac{q_1(x)}{q_0}\sin^2(\pi x)-8\sqrt{2}\pi^3\frac{q_1'(x)}{q_0}\sin(2\pi x)\\ -4\pi^2\frac{q_1''(x)}{q_0}\sin^2(\pi x)\Big]dx+\dots\,. \tag{3.40}$$

In the following tables, we present the eigenvalues of perturbed problem in cases of concrete and steel beams with hinged both ends and we note that in each case, we take $q_1(x)=10^{10}x$ as a linear variation and so we have $q_1'(x)=10^{10}$ and $q_1''(x)=0$, so we have:

Table 4.1: ***Concrete eigenvalues of vibrating non-homogenous hinged beam***

The Value of ε	The Eigenvalues $\lambda(\varepsilon)$
$\varepsilon=0$	$4\pi^4$
$\varepsilon=0.1$	1.3719×10^{10}
$\varepsilon=0.2$	2.75037×10^{10}
$\varepsilon=0.3$	4.12556×10^{10}

Table 4.2: ***Steel eigenvalues of vibrating non-homogenous hinged beam***

The Value of ε	The Eigenvalues $\lambda(\varepsilon)$
$\varepsilon=0$	$4\pi^4$
$\varepsilon=0.1$	1.11325×10^{9}
$\varepsilon=0.2$	2.22649×10^{9}
$\varepsilon=0.3$	3.33974×10^{9}

Thus, Equation (4.35) equipped with the hinged boundary conditions given by (4.35i) can be solved for $u_\varepsilon(x,0)$ and the unique solution $w(x)$ that is orthogonal with $e_n(x)$ is given by (Ivar Stakgold [14]) as:

$$w(x)=\sum_{m\neq n}\Big[\frac{-1}{(m^4-n^4)\pi^4}\langle 8n^4\pi^4\frac{q_1(x)}{q_0}\sin(n\pi x)-8\sqrt{2}n^3\pi^3\frac{q_1'(x)}{q_0}\cos(n\pi x) \\ -4n^2\pi^2\frac{q_1''(x)}{q_0}\sin(n\pi x),2\sin(m\pi x)\rangle 2\sin(m\pi x)\Big]. \tag{4.41}$$

4.4 An Unperturbed Non-Linear Problem

In this Section, we give a brief application of Green's function obtained for the fourth-order operator $\frac{d^4}{dx^4}$ to a non-linear eigenvalue problem. We find an estimate of the range within which a non-linear eigenvalue problem lies.

We consider the following problem:

$$\begin{cases} \dfrac{d^4u(x)}{dx^4}=\alpha\sin(u(x)),\ 0<x<1, \\ \textit{with hinged boundary conditions}, \\ u(0)=0, u''(0)=0, u(1)=0, u''(1)=0. \end{cases} \tag{4.42}$$

The linearization of the given problem (4.42) about the trivial solution $u\equiv 0$ gives us the following linearized problem:

$$\begin{cases} \dfrac{du^4(x)}{dx^4}=\alpha u(x),\ 0<x<1, \\ \textit{with hinged boundary conditions}, \\ u(0)=0, u''(0)=0, u(1)=0, u''(1)=0. \end{cases} \tag{4.43}$$

In fact, we have found earlier the eigenvalues and the normalized eigenfunctions for (4.43) which are $\alpha_n=4n^4\pi^4, e_n(x)=\sqrt{2}\sin(n\pi x), n=1,2,3,...$ respectively. As zero is not an eigenvalue, we can construct Green's function $G(x,\xi)$ satisfying the following:

$$\begin{cases} \dfrac{d^4G(x,\xi)}{dx^4} = \delta(x-\xi), \ x \neq \xi, \quad 0 < x < 1, \\ \text{with hinged boundary conditions}, \\ G(0,\xi) = 0, G''(0,\xi) = 0, G(1,\xi) = 0, G''(1,\xi) = 0. \end{cases} \tag{4.44}$$

Earlier, we have found Green's function of (4.44) see Eq. (2.14), which is:

$$G(x,\xi) = \begin{cases} \dfrac{1}{6}\xi(1-\xi)x + \dfrac{1}{6}(\xi-1)x^3, & 0 \le x < \xi \le 1, \\ \dfrac{1}{3}\xi(1-\xi)(1-x) + \dfrac{1}{6}\xi(x-1)^3, & 0 \le \xi < x \le 1. \end{cases} \tag{4.45}$$

Thus, problem (4.42) can be written as the equivalent integral equation as following:

$$\lambda u(x) = \int_0^1 G(x,\xi)\sin(u(\xi))d\xi = Au\,, \tag{4.46}$$

where $\lambda = \dfrac{1}{\alpha}$ and A is a non-linear Hammerstein integral operator whose linearization at $u \equiv 0$ is a linear operator B with kernel $G(x,\xi)$, so the problem:

$$Bu = \lambda u\,, \tag{4.47}$$

which can be written as the following integral equation:

$$\lambda u(x) = \int_0^1 G(x,\xi)u(\xi)d\xi\,.$$

Problem (4.47) has the eigenvalues $\lambda_n = \dfrac{1}{\alpha_n}$ and the normalized eigenfunctions $e_n(x) = \sqrt{2}\sin(n\pi x), n = 1,2,3,\dots$.

The norm of the problem (4.47), gives us:

$$|\lambda|\|u\| = \|\lambda u\| = \|Bu\| \le \|B\|\|u\|,$$

so we have $\|B\|$ is the largest eigenvalue $\lambda_1 = \dfrac{1}{4\pi^4}$ of the linear integer operator B. Therefore, we can say that the problem (4.46) can have a non-trivial solution only if $0 < \lambda \le \dfrac{1}{4\pi^4}$ i.e. $\alpha \ge 4\pi^4$.

4.5 The Liapunov-Schmidt Method for Non-Linear Problem (Ivar Stakgold[14])

The operator A has the property $A0 = 0$ and $A'0 = B$. We set $u_0 = 0$ and $r = Ru$, to give:

$$\lambda u = Au = Bu + Ru\,, \quad \lim_{u \to 0} \frac{\|Ru\|}{\|u\|} = 0\,, \qquad (4.48)$$

where $Bu = \int_0^1 G(x,\xi) u(\xi) d\xi, Ru = \int_0^1 G(x,\xi)[\sin(u(\xi) - u(\xi)] d\xi$.

We can split the remainder Ru into a homogenous term Cu (say) of third degree and a higher degree term Du.

$$Ru = Cu + Du = -\int_0^1 G(x,\xi) \frac{u^3(\xi)}{6} d\xi + \int_0^1 G(x,\xi) \left[\sin(u(\xi)) - u(\xi) + \frac{u^3(\xi)}{6} \right] d\xi\,. \qquad (4.49)$$

If we set $\delta = \lambda - \lambda_n$, $u = ce_n + w$, $\langle w, e_n \rangle = 0$ for some constant c, then problem (4.48) becomes:

$$Bu - \lambda_n u = \delta u - Ru\,. \qquad (4.50)$$

Problem (4.48) is consistent if and only if $\langle \delta u - Ru, e_n \rangle = 0$.

The order information of w and the specific forms of C and D enable us to write δ_c to the first-order as following:

$$\delta_c = \langle C(ce_n, e_n) \rangle = c^3 \langle Ce_n, e_n \rangle . \qquad (4.51)$$

The definition of the operator C in (4.49), gives us:

$$\langle Ce_n, e_n \rangle = -\int_0^1 e_n(x)dx \int_0^1 G(x,\xi) \frac{e_n^{\ 3}(\xi)}{6} d\xi = \frac{-1}{4n^4\pi^4}, \quad n = 1,2,..., \qquad (4.52)$$

so we have $\delta = \lambda - \lambda_n = \dfrac{-c^2}{4n^4\pi^4}, n = 1,2,...$. Therefore, we have:

$$\alpha - \alpha_n = \frac{1}{\lambda} - \frac{1}{\lambda_n} = -\frac{\delta}{\lambda^2{}_n} = 4c^2 n^4 \pi^4, \quad n = 1,2,... \ . \qquad (4.53)$$

Equation (4.53) gives us an estimate of the difference between the eigenvalues α of the non-linear problem (4.42) and the eigenvalues $\alpha_n, n = 1,2,...$ of the linearized problem (4.43) (see Ivar Stakgold [14]).

4.6 A Perturbed Non-Linear Problem

Consider the following problem:

$$\begin{cases} \dfrac{d^4u}{dx^4} + \varepsilon \dfrac{d^2}{dx^2}\left(\dfrac{q_1(x)}{q_0} \dfrac{d^2u}{dx^2} \right) = \alpha \sin(u), \quad 0 < x < 1, \\ \textit{with hinged boundary conditions}, \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \qquad (4.54)$$

The linearization of the given problem (4.54) about the identical solution $u \equiv 0$ gives us the following linearized problem:

$$\begin{cases} \dfrac{d^4u}{dx^4} + \varepsilon\left[\dfrac{q_1(x)}{q_0}\dfrac{d^4u}{dx^4} + 2\dfrac{q_1'(x)}{q_0}\dfrac{d^3u}{dx^3} + \dfrac{q_1''(x)}{q_0}\dfrac{d^2u}{dx^2}\right] = \alpha u\,, \quad 0 < x < 1, \\ \textit{with hinged boundary conditions}, \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \tag{4.55}$$

In fact, we have found early the eigenvalues and the corresponding normalized eigenfunctions for (4.55) which are $\alpha_{linear} = 4n^2\pi^2 + \varepsilon\alpha_{\varepsilon}(0), n = 1,2,3,...$ and $u_n(x,\varepsilon) = e_n(x) + w(x)$ (taken by Ivar Stakgold [14]) respectively, where $\alpha_{\varepsilon}(\varepsilon)$ and $w(x)$ are given in Eq. (4.39) and (4.41) respectively. As zero is not an eigenvalue, we can construct Green's function $G_{\varepsilon}(x,\xi)$ satisfying the following:

For all $x \neq \xi,\ \ 0 < x < 1$, we have:

$$\begin{cases} \dfrac{d^4G(x,\xi)}{dx^4} + \varepsilon\left[\dfrac{q_1(x)}{q_0}\dfrac{d^4G(x,\xi)}{dx^4} + 2\dfrac{q_1'(x)}{q_0}\dfrac{d^3G(x,\xi)}{dx^3} + \dfrac{q_1''(x)}{q_0}\dfrac{d^2G(x,\xi)}{dx^2}\right] = \delta(x-\xi), \\ \textit{with hinged boundary conditions}, \\ G(0,\xi) = 0, G''(0,\xi) = 0, G(1,\xi) = 0, G''(1,\xi) = 0. \end{cases} \tag{4.56}$$

The Green's function of the problem (4.56) is given by:

$$\begin{aligned} G_{\varepsilon}(x,\xi) = -\varepsilon\int_0^1 \Big[&\frac{q_1(x)}{q_0}\frac{d^4G(x,\xi)}{dx^4} + 2\frac{q_1'(x)}{q_0}\frac{d^3G(x,\xi)}{dx^3} \\ &+ \frac{q_1''(x)}{q_0}\frac{d^2G(x,\xi)}{dx^2}\Big] G(x,\xi)d\xi, \end{aligned} \tag{4.57}$$

where $G(x,\xi)$ is Green's function of:

$$\frac{d^4G(x,\xi)}{dx^4} = \delta(x-\xi), x \neq \xi, \quad 0 < x < 1.$$

Thus problem (4.54) can be written as the equivalent integral equation as:

$$\lambda u(x) = \int_0^1 G_\varepsilon(x,\xi)\sin(u(\xi))d\xi = Au\,, \qquad (4.58)$$

where $\lambda_{NL} = \dfrac{1}{\alpha_{NL}}$ and A is a non-linear Hammerstein integral operator whose linearization at $u \equiv 0$ is a linear operator B with kernel $G_\varepsilon(x,\xi)$. Thus, the equation:

$$Bu = \lambda u\,, \qquad (59)$$

which can be written as the following integral equation:

$$\lambda u(x) = \int_0^1 G_\varepsilon(x,\xi)u(\xi)d\xi\,.$$

Problem (59) has the eigenvalues $\lambda_L = \dfrac{1}{\alpha_L} = \dfrac{1}{4n^4\pi^4 + \varepsilon\alpha_\varepsilon(0)}$, $n = 1,2,\ldots$ and the normalized eigenfunction $u_n(x,\varepsilon) = e_n(x) + \varepsilon w(x)$ (given by Ivar Stakgold [14]), where $\alpha(\varepsilon)$ and $w(x)$ are given in Eq. (4.39) and (4.41) respectively. The norm of the given problem in Eq. (4.59) gives us:

$$|\lambda|\|u\| = \|\lambda u\| = \|Bu\| \leq \|B\|\|u\|,$$

so we have $\|B\|$ is the largest eigenvalue λ_L. Therefore, we can conclude that problem (4.58) can have a non-trivial solution only if $0 < \lambda_{NL} \le \dfrac{1}{4\pi^4 + \varepsilon\alpha_\varepsilon(0)}$ and

$$\alpha = \alpha_{NL} = \frac{1}{\lambda_{NL}} \Rightarrow 4\pi^4 + \varepsilon\alpha_\varepsilon(0) \le \alpha.$$

Illustration Example:

We suppose that $q_1(x) = 10^{10}x$ as a linear variation and so we have $q_1'(x) = 10^{10}$ and $q_1''(x) = 0$. In the following tables, we present the eigenvalues of perturbed non-linear problem in cases of concrete and steel beams with hinged both ends, we have:

Table 4.3: ***Concrete eigenvalues of vibrating non-linear beam with hinged ends***

The Value of ε	The Eigenvalues of Problem (4.58) $\lambda = \lambda_{NL} \le$	The Eigenvalues of Problem (4.54) $\alpha = \alpha_{NL} \ge$
0	$\frac{1}{4\pi^4}$	$4\pi^4$
0.1	7.2892×10^{-11}	1.3719×10^{10}
0.2	3.6359×10^{-11}	2.75037×10^{10}
0.3	2.4239×10^{-11}	4.12556×10^{10}

Table 4.4: ***Steel eigenvalues of vibrating non-linear beam with hinged ends***

The Value of ε	The Eigenvalues of Problem (4.58) $\lambda = \lambda_{NL} \le$	The Eigenvalues of Problem (4.54) $\alpha = \alpha_{NL} \ge$
0	$\frac{1}{4\pi^4}$	$\frac{1}{4\pi^4}$
0.1	8.9827×10^{-10}	1.11325×10^{9}
0.2	4.4914×10^{-10}	2.22649×10^{9}
0.3	2.9942×10^{-10}	3.33974×10^{9}

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In this study, we considered the fourth-order boundary value problem arising from the Euler-Bernoulli model of an elastic properties of the beam held under deferent supports, that is a 4^{th} order boundary value problem arising from non-homogenous Euler Bernoulli model in both cases static and dynamic have been studied. We described some basic spectral properties of the resulting fourth-order deferential linear operator and constructed Green's function for three sets of boundary conditions. Using Green's function and perturbation approximation, the effect of inhomogeneity on the deflection of the beam and normalized eigenfrequency have also been found, correct to the 1^{st} order. In addition, we considered the dynamic problem of transverse vibrations and obtained eigenvalues and normalized eigenfunctions that correspond to these eigenvalues.

The focus of this thesis has been studied these problems for Euler-Bernoulli beam with variable elastic properties. We used kind of the perturbations formulation in which both elastic and dynamic model have been discussed. In static case, Green's function has been used to find deflection in a non-homogeneous beam correct to first-order. These results have been graphically presented for two interested cases:

i. Concrete beam.

ii. Steel beam.

The eigenvalue problem arising from transverse vibrations of non-homogeneous beams has then been studied using perturbation approach. The smallest eigenvalue representing the eigenfrequency of the vibrating beam has been obtained.

A non-linear eigenvalue problem for non-homogenous elastic properties of beam has been studied to find the variation in eigenfrequency due to non-homogeneity, following Liapunov-Schmidt method for non-linear problem has also been studied to find the variation in the eigenvalue due to non-linear forcing term.

5.2 Recommendations for Future Work

We have considered the three sets of boundary conditions to solve our fourth-order unperturbed and perturbed boundary value problems. More sets of boundary value problems can be considered. Also, non-homogeneous boundary conditions, such as a force being applied at one end, may be considered.

In the final part of this thesis, a direction in the non-linear eigenvalue problems involving the fourth-order differential linear operator has been painted out. In future work, we intend to explore this further to know more about bifurcation of eigenvalues and normalized eigenfunctions. Moreover, it is intended to study the bifurcation of eigenvalues for non-linear perturbed problem.

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